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Numerical Solution of Fractional Stochastic Integro-Differential Equations by the Operational Tau Method

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1. Abstract

Stochastic Differential Equations (SDEs) driven by a Brownian motion or a more general Levy process arise in many applications for the description of random-phenomena treated in physics, engineering, economics and mathematical finance.

In recent decades, the subject of stochastic differential equations is gaining much impact and attention [1,49] and surveys [6,41,50,51] with references therein. Moreover, the study of numerical solutions of stochastic differential equations have been progressed considerably in the last few years [18,25,50,52-54]. In [41] some discrete approximation methods are explained for solving stochastic differential equations, with additional requirements [54]. Misawa [42] proposed the formulation of composition methods for solving stochastic differential equations. Li et al [53,54] proposed explicit numerical approximations for stochastic differential equations in finite and infinite horizons. Efficient methods based on Runge-Kutta methods were proposed [29,32,39,44,55-59].

Fractional calculus provides a powerful tool to model various phenomena in viscoelasticity, fluid mechanics, biology, chemistry, control theory and other areas of science. Mathematical folklore sets the birth of the concept of fractional calculus in the year 1695 by the answer to a question raised by L'Hôpital (1661-1704) to Leibniz (1646-1716), in which he sought the meaning of Leibniz's notation $\frac{dy_n}{dx}$ for derivatives if$n=\frac{1}{2},\frac{1}{3}$, ….In this reply, date 30 September 1695, Leibniz wrote to L'Hôpital " This is an apparent paradox from which, one day, useful consequence will be drawn…".

Spectral methods have become increasingly popular in recent years, there are three most commonly used spectral versions, namely the Galerkin-type, Tau and collocation methods. Among these is the Tau method extensively applied for the numerical solution of differential equations [60-65]. It was introduced by Lanczos [66] in 1938 for solving ordinary differential equations. In 1981, Ortiz and Samara [67] presented a new approach to the Tau method by proposing an operational technique for the numerical solution of a single nonlinear ordinary differential equation with some supplementary conditions. Cruz-Santiago et al [68] showed later that it is not necessary to employ a complicated reformulation of this Lanczos process to construct polynomial solutions of ordinary differential equations.

During the last thirty years' considerable work has been done in the development of the Tau method technique in theoretical analysis and numerical applications [61,62] for numerical solution of many problems such as integro-differential equations [62,69,70] etc.

The main objective of the present work is to develop an operational approach of Tau method for solving fractional stochastic integro-differential equations. The main characteristic behind the approach using this technique is that it can transform the problems into a system of algebraic equations wherein its solution is easy. We chose the (n-m+1) nodes of the shifted Chebyshev polynomials as suitable collocation nodes. This method provides an accurate numerical solution of fractional stochastic integro-differential equations and the accuracy of the proposed method is demonstrated by several examples.

The article is structured as follows. In section 2, we introduce some essential definition and preliminary notations. In section 3, we study the operational Tau method. The operational Tau method to approximate the equation (6.1) below, is utilized in section 4. In section 5, an error estimation and a scheme of convergence of proposed method is presented. Section 6 is devoted to some numerical examples.

4. Basic Definitions

In this section, we state some basic definitions, notations and fundamental assumptions, as needed in the sequel. There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. the Riemann- Liouville and the Caputo approach.

4.1. Definition: The Riemann-Liouville fractional integral operator of order θ ($\theta > 0$) is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \quad \alpha, t > 0, \qquad (4.1)$$

$$J^{0} f(t) = f(t). \qquad (4.2)$$

Here Γ is the Gamma function. Some of the most important properties of the operator $J\theta$ for $f(t)$, are as follows

i) $J^{\alpha} J^{\theta} f(t) = J^{\alpha+\theta} f(t)$

ii) $J\alpha t\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$

iii) $J^{\theta} J^{\alpha} f(t) = J^{\alpha} J^{\theta} f(t)$.

4.2. Definition: The Caputo fractional derivatives of order θ of $f(t)$ are defined as

$$D^{\alpha} f(t) = J^{n-\alpha} D^{n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} f(s)\,ds, \; t > 0, \; n-1 < \alpha < n. \qquad (4.3)$$

for Dn is the classical differential operator of order n. Also

$$D^{\alpha} t^{\mu} = \begin{cases} 0 & \mu \in N_0, \mu < \lceil \alpha \rceil \\ \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)} t^{\mu-\alpha}, \mu \in N_0, \mu \geq \lceil \alpha \rceil \text{ or } & \mu \in N, \mu > \lfloor \alpha \rfloor, \end{cases} \qquad (4.4)$$

for $N = 1,2, \ldots$ and $N0 = 0,1,2, \ldots$, while $\lceil \alpha \rceil$ and $\lfloor \alpha \rfloor$ are the floor and ceiling function, respectively.

4.3. Definition: The well-known Chebyshev polynomials $T_n(x)$, defined on the interval I = [-1,1] have the following properties

$$T_0(x) = 1, T_1(x) = x, \ldots, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

In order to use these polynomials on the interval $x \in$ [0, L] we define the so-called shifted Chebyshev polynomials by introducing the change of variable $t = \frac{2x}{L} - 1$. Let $T_n^*(x)$ denote L the shiftedChebyshev polynomials obtained as follows

$$T_{n+1}^*(x) = 2\left(\frac{2x}{L} - 1\right) T_n^*(x) - T_{n-1}^*(x), i = 1,2, \ldots$$

(4.5)

Where $T_0^*(x) = 1$ and $T_1^*(x) = \frac{2x}{L} - 1$.

4.4. Definition: Let (Ω, F, P) be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The Brownian motion B is a stochastic process with the following properties [51]:

1. $B(0) = 0$.
2. $B(t) - B(s)$, $t > s$ is independent of $\mathcal{F}_s$. That means for $0 < u < v < s < t < \infty$, the increments $B(t) - B(s)$ and $B(v) - B(u)$ are independent.
3. $B(t) - B(s)$ is normal distributed with mean 0 and variance $t - s$.
4. $B(t), t \geq 0$ are continuous functions of t.

4.5.Definition: Anintegral of the form $\int_a^b f(t)dB(t)$ where $B(t)$ denotes the Brownian motion process can be defined in various ways. This integral can be approximated as

$$\sum_{i=1}^{n} f(T_i)(B(t_i) - B(t_{i-1})) \cdot \qquad (4.6)$$

where τi is chosen arbitrary in the subinterval [t_{i-1}, t_i] of the partition $a = t_0 < t_1 < \ldots < t_n = b$. If τ_i is taken as t_{i-1}, (4.6) is an Itô integral [4,47,71,72] and if $T_i = \frac{t_{i-1}+t_i}{2}$, (4.6) is a Stratonovich integral[12,55,73].

5. The Operational Tau Method

In this section, we state some relevant properties of the operational Tau method. Let $\varphi(t)$ and $\psi(t)$ be integrable functions on $[a, b]$, we define the scalar product <, > by

$$< \varphi(t), \psi(t) >_{\omega} = \int_a^b \varphi(t)\psi(t)\omega(t)\,dt$$

where $\| \varphi \|_{\omega}^2 = < \varphi(t), \varphi(t) >_{\omega}$ and $\omega(t)$ is a weight function.

Let $L_{\omega}^2[a, b]$ bethespaceof all functions $f: [a, b] \to \mathbb{R}$ with $\| f \|_{\omega}^2 < \infty$. The main idea of the method is to approximate $f(t) \in L_{\omega}^2[a, b]$. Let us consider

$$f(t) = \sum_{i=0}^{\infty} f_i t^i = \bar{f} T_t \qquad (5.1)$$

Where $\bar{f} = [f_i]_{i=0}^{\infty}$ is a vector of unknown coefficients and $T_t = [1, t, t^2, \ldots]^*$ where denotes the transpose. In practice, only the first (m+1)-terms are considered [61,63,74,75].

The Tau method is designed to convert linear or non-linear differential equations, delay differential equations or a system of these equations to a system of linear algebraic equations based on three simple matrices

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & \vdots 0 & 0 \ddots & \end{bmatrix}, \eta = \begin{bmatrix} 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \\ 0 & 0 & \vdots 3 & 0 \ddots & \end{bmatrix}, p = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 & \\ 0 & 0 & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \\ 0 & 0 & 0 & 0 & \end{bmatrix}.$$

In the sequel we need operators on polynomials, stated in a lemma as follows.

5.1. Lemma

Suppose that $f(t)$ is a polynomial of the form $f(t) = \sum_{i=0}^{\infty} f_i t^i = \bar{f} T_t$, then we have

$$D^r f(t) = \frac{d^r}{dt^r} f(t) = \bar{f} \eta^r T_t \qquad r=0,1,2,3,\ldots \qquad (5.2)$$

$$t^s f(t) = \bar{f} \mu^s T_t \qquad s=0,1,2,3,\ldots \qquad (5.3)$$

$$\int_a^x f(t)dt = \bar{f} p T_x - \bar{f} p T_a \qquad p \text{ is matrix } p \qquad (5.4)$$

For a proof see [63]

6. Analysis of the Solution Method

The purpose of this section is to analyze a numerical solution of the following fractional stochastic integro-differential equation

$$D^\alpha f(t) = g(t) + \int_0^t k_1(t,s) f(s) ds + \int_0^t k_2(t,s) f(s) dB(s),$$

$$t \in [0,1], \qquad (6.1)$$

by using the operational Tau method. Where $g(t)$, $k_1(t,s)$ and $k_2(t,s)$ are given functions on the same probability space (Ω,F,P), $f(t)$ is the unknown function and $B(t)$ is a Brownian motion also, $t,s \in [0,1]$ are variables.

In order to use the proposed method, we pursue the following process. By equation (4.3), (5.1) and (5.4), we first approximate $D^\alpha f(t)$ as [74]

$$D^\alpha f_n = J^{n-\alpha} D^n(\bar{f} T_t) = J^{n-\alpha}(\bar{f} \eta^n T_t) = \bar{f} \eta^n J^{n-\alpha}(T_t) \qquad (6.2)$$

By definition (3.1) and relation (iii) we obtain:

$$J^{n-\alpha}(T_t) = [J^{n-\alpha}(1), J^{n-\alpha}(t), \ldots, J^{n-\alpha}(t^\gamma), \ldots]^* = [\frac{\Gamma(1) t^{n-\alpha}}{\Gamma(n-\alpha+1)}, \frac{\Gamma(2) t^{n-\alpha+1}}{\Gamma(n-\alpha+2}, \ldots, \frac{\Gamma(\gamma+1) t^{n-\alpha+\gamma}}{\Gamma(n-\alpha+\gamma+1)}, \ldots]^* = \acute{\Gamma}\Pi \qquad (6.3)$$

Where, $\Gamma' = \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(n-\alpha+1)} & 0 & \cdots & 0 & \cdots \\ 0 & \frac{\Gamma(2)}{\Gamma(n-\alpha+2)} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(\gamma+1)}{\Gamma(n-\alpha+\gamma+1)} & \cdots \\ 0 & 0 & 0 & 0 & \ddots \end{bmatrix}$ is a diagonal matrix

and $\Pi = [t^{n-\alpha}, t^{n-\alpha+1}, \ldots, t^{n-\alpha+\gamma}, \ldots]^*$.

Then, by substituting equation (6.3) in equation (6.2), we have

$$D^\alpha f_n(t) \simeq \bar{f} \eta^n \acute{\Gamma} \Pi. \qquad (6.4)$$

The function of $g(t)$ can be approximated as,

$$g_n(t) \simeq \sum_{i=0}^{\infty} g_i t^i = \bar{g} T_t \qquad (6.5)$$

Where, $\bar{g} = [g_0, g_1, g_2, \ldots]$.

In order to approximate $\int_0^t k_1(t,s) f(s) ds$, first of all, we approximate $k_1(t,s)$ by the Tau method,

$$k_{1,n}(t,s) = \sum_{p=0}^{n} \sum_{q=0}^{n} k_{pq} t^p s^q = T_s \bar{k}_1 T_t \qquad (6.6)$$

Where $\bar{k}_1 = \begin{bmatrix} k_{00} & k_{01} & k_{02} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ k_{n0} & k_{n1} & k_{n2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, T_s - [1, s, s^2, \ldots]$.

Substituting equation (6.6) and (5.1) yields [62]

$$\int_0^t k_{1,n}(t,s) f_n(s) ds \simeq \bar{f} k_1 T_t \qquad (6.7)$$

Where,

$$K_1 = \begin{bmatrix} 0 & k_{00} & k_{10}+\frac{1}{2}k_{01} & k_{20}+\frac{1}{2}k_{11}+\frac{1}{3}k_{02} & k_{30}+\frac{1}{2}k_{21}+\frac{1}{3}k_{12}+\frac{1}{4}k_{03} & \cdots \\ 0 & 0 & \frac{1}{2}k_{00} & \frac{1}{2}k_{10}+\frac{1}{3}k_{01} & \frac{1}{2}k_{20}+\frac{1}{3}k_{11}+\frac{1}{4}k_{02} & \cdots \\ 0 & 0 & 0 & \frac{1}{3}k_{00} & \frac{1}{3}k_{10}+\frac{1}{4}k_{01} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{4}k_{00} & \cdots \\ 0 & 0 & 0 & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Now we approximate the second integral in equation (5.1) by approximating first the $k2(t,s)$ by the Tau method,

$$k_{2,n}(t,s) = \sum_{p=0}^{n} \sum_{q=0}^{n} k_{pq} t^p s^q = T_s \bar{k}_1 T_t \qquad (6.8)$$

Where, $\bar{k}_2 = \begin{bmatrix} k_{00} & k_{01} & k_{02} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ k_{n0} & k_{n1} & k_{n2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

Employing equations (4.6), (5.1) and (6.8) in the second integral yields,

$$\int_0^t k_{2,n}(t,s) f_n(s) dB(s) \bar{f} \bar{k}_2 \bar{B}, \qquad \bar{B} = [\Delta B_1, \Delta B_2, \ldots]^T. \quad (6.9)$$

Where $\Delta B_i = B(\varphi_i) - B(\varphi_{i-1})$ in the subinterval $[\varphi_{i-1}, \varphi_i]$ of the partition $a = \varphi_0 < \varphi_1 < \cdots < \varphi_n = b$ in which the Brownian motion has normal distribution [72] as follows $B(\varphi) = \sqrt{\varphi} N(0,1)$

By substituting equations (5.4), (5.5), (5.7), (5.9) in equation (5.1) we have,

$$\bar{f} \eta^m \acute{\Gamma} \Pi = \bar{g} T_t + \bar{f} K_1 T_t + \bar{f} \bar{k}_2 \bar{B}. \qquad (6.10)$$

We collocate equation (5.10) at $(m+1)$ points $t_\delta = s_\delta$, $\delta = 0,1,2,\ldots,m-r+1$ as

$$\bar{f} \eta^m \acute{\Gamma} \Pi = \bar{g} T_\delta + \bar{f} K_1 T_\delta + \bar{f} \bar{k}_2 \bar{B} \qquad (6.11)$$

For suitable collocation points, we use roots of shifted Chebyshev polynomials $T^*_{(m-r+1)}(t)$. Equation (4.11), together with r equations of initial and boundary conditions give $(m+1)$ equations.

7. Error Analysis

In this section, an error estimation for the approximate solution is provided. In [72], Taheri at al proposed a convergence analysis for solving SIFDEs by spectral collocation method based on the shifted Legendre polynomials. They applied the error function which we describe it in the following to obtain an upper bound for the estimated error. Also, in [76], Mirzaee and Samadyar discussed an error estimation of the method based on strictly positive definite RBFs for fractional stochastic integro-differential equations.

We expand a general scheme to estimate the error of our method from [77]. Let us write the residual $R(x)$ of equation (5.1) as

$$R(x) = \bar{f}\eta^n \acute{\Gamma}\Pi - \bar{g}T_t - \bar{f}K_1T_t - \bar{f}\bar{k}_2\bar{B} \tag{7.1}$$

Which $R(x) \in R_s = spain_{i\in s}\{x^i\}$.

To obtain an error estimation for the approximate solution of equation (6.1) with supplementary conditions, let us call $e_n(t) = f(t) - f_n(t)$ the error function of the Tau approximants $f_n(t)$ to $f(t)$ is the exact solution of equation (6.1). Therefore, $f_n(t)$ satisfies the following equation

$$D^\alpha f_n(t) = g_n(t) + \int_0^t k_{1,n}(t,s) f_n(s)ds + \int_0^t k_2(t,s) f_n(s)\,dB(s) + H_n(t), \quad t \in [0,1],$$
(7.2)

Which $H_n(t)$ is the perturbation term. $H_n(t)$ is chosen in such a way that the residuals of $D^\alpha f_n(t)$ match the component of $H_n(t)$ belonging to R_s.

Now, by subtracting equation (6.2) from equation (5.1) we have

$$D^\alpha(f(t) - f_n(t)) = (g(t) - g_n(t)) + \int_0^t k_1(t,s)f(s) - k_{1n}(t,s)f_n(s)\,ds + \int_0^t (k_2(t,s)f(s) - k_2(t,s)f_n(s))dB(s) - H_n(t), t \in [1,0],$$

or

$$D^\alpha(e_n(t)) = \int_0^t k_1(t,s)e_n(t)\,ds + \int_0^t (k_2(t,s)e_n(t)\,dB(s)H_n(t), t \in [1,0]$$
(7.3)

So, we can rewrite equation (6.3) in the following form

$$e_n(t) = I^\alpha\left(\int_0^t k_1(t,s)e_n(s)ds\right) + I^\alpha\left(\int_0^t k_2(t,s)e_n(s)dB(s)\right) - I^\alpha H_n(t).$$
(7.4)

So, we can write

$$e_n(t) = \mathrm{E}_1 + \mathrm{E}_2 - \mathrm{E}_3, \tag{7.5}$$

To estimate $E1$, $E2$ and $E3$, we state some properties that can be found in [79,80].

7.1. Definition: Sobolev pace on $I = [0,1]$, $H^m(I)$ is a Hilbert space with the norm as

$$\| f \|_{H^m(I)} = \left(\sum_{i=o}^m \| f^i \|^2_{L^2(I)}\right)^{\frac{1}{2}}.$$

7.1.1. Lemma: (Sobolev inequality) Assume (a,b) be a bounded interval of the real line for any function $f \in H_1(a,b)$ the following inequality holds So,

$$\| f \|_{L^\infty} \le \left(\frac{1}{b-a} + 2\right)^{\frac{1}{2}} \| f \|^{\frac{1}{2}}_{L^2(a,b)} \| f \|^{\frac{1}{2}}_{H^1(a,d)}$$

from [79] we have

$$\| E_1 \|_{L^2(I)} \le \frac{1}{\Gamma(\alpha)} \max_{s,t\in I} |k_1(s,t)| \, \| f(t) - f_n(t) \|_{L^2(I)}, \tag{7.6}$$

for the E_2 from [79] and [80] we have

$$\| E_1 \|_{L^2(I)} \le \frac{1}{\Gamma(\alpha)} \max_{s,t\in I} |k_2(s,t)| (L_1 \| f(t) - f_n(t) \|_{L^2(I)} + L_2 \| B(t) \|_{L^2(I)}, \tag{7.7}$$

which L_1 and L_2 are Lipschitz constants and

$$E_3 = \frac{1}{\Gamma(\alpha)} \| H_n(t) \| \tag{7.8}$$

The proof of (7.6)- (7.8) can be found in [79-80].

If we consider two successive polynomial approximations $f_n(t)$ and $f_{n+1}(t)$ then the maximum of $e_n(t)$ is obviously greater than (or equal to) that of $e_{n+1}(t)$, so we have

$$e_n(t) - e_{n+1}(t) = f_{n+1}(t) - f_n(t)$$

7.2. Definition: Given two expansion $f_i(t)$ and $f_j(t)$, define $\mu_{i,j}$ to be the maximum deviation of one from the other, i.e. $\mu_{i,j} = \max_{t\in[0,1]} |f_i(t) - f_j(t)| = \| f_i - f_j \|$.

7.3. Definition: For the error functions $e_i(t)$ and $e_j(t)$, where $j>i$, define the parameter $\rho_{i,j}$ to be

$$\rho_{i,j} = \frac{\| e_j(t) \|}{\| e_i(t) \|}.$$

where $ei(t)$ and $ej(t)$ are obtained by equation (7.5). It follows that $\rho_{i,j} \le 1$ and $\mu_{i,j} = \| e_i - e_j \|$ we have the following proposition.

7.3.1 Proposition: Suppose that n is given, choose an integer $m \ge 1$ such that $\mu n,n+m$ is negligible compared to unity, then an upper for the maximum error $\|en\|$ is obtained as $e_n \le \frac{\mu_{n,n+m}}{1-\rho_{n,n+m}}$.

We continued our method while the condition $|f_{n+1}(t) - f_n(t)| < \varepsilon$ satisfied for a preassigned positive ε.

8. Numerical Results and Comparison

In this section some numerical examples are given to illustrate the high accuracy of the method. We present also a comparison between our described technique and the proposed method in [72].

8.1. Example

Consider the following fractional stochastic integro-differential equation [72,76]

$$D^{\alpha}f(t) = -\frac{t^5 e^t}{5} + \frac{6t^{2.25}}{\Gamma(3.25)} + \int_0^t e^t s f(s)\,ds + \sigma \int_0^t e^t\, s f(s)\,dB(s)$$

, $t \in [0,1]$

Where the exact solution with $\theta = 0.75$, $\sigma = 0$ is $f(t) = t^3$.

We have solved this problem for $\alpha = 0.75$, $\sigma = 0$, by applying the technique described in section 4 with $m = 3$. (Figure 1) shows the numerical solution for $\alpha = 0.75$ and $\sigma = 0$.

Here we have, Now, by applying Eq (20) we have,

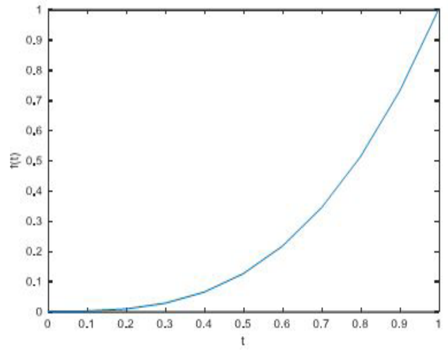

Figure 1: The graph of the approximate solution of example 1 for $\alpha = 0.75$, $\sigma = 0$ with m=3

$$\Gamma\Pi = \left[\frac{\Gamma(1)t^{\frac{1}{4}}}{\Gamma(\frac{5}{4})}, \frac{\Gamma(2)t^{\frac{5}{4}}}{\Gamma(\frac{9}{4})}, \frac{\Gamma(3)t^{\frac{9}{4}}}{\Gamma(\frac{13}{4})}, \frac{\Gamma(4)t^{\frac{13}{4}}}{\Gamma(\frac{17}{4})}\right]^{*}, \bar{K}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, K_1 = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{g} = [0, 0, \frac{6t^{\frac{1}{4}}}{\Gamma(3.25)}, 0].$$

$1.37025f_0 - 0.647601f_1 - 1.68156f_2 - 2.15665f_3 + 2.15665 = 0$

$0.64308f_0 - 0.84081f_1 - 1.11271f_2 - 1.02562f_3 + 1.02562 = 0$

$0.110005f_0 - 0.807653f_1 - 0.406181f_2 - 0.167184f_3 + 0.167184 = 0$

$0.00147926f_0 - 0.4874f_1 - 0.0297136f_2 - 0.00150945f_3 + 0.00150945 = 0$

And from this obviously follows

$\bar{f} = [0,0,0,1], \bar{f} \times T_t = t^3$

which is the exact solution.

The behaviors of approximate solutions using the proposed method for $\sigma = 1, \alpha = 0.75$ with $m = 8,11$ are presented in (Figures 2).

Furthermore, to make a comparison, you can see the approximate solution by the presented method in [72].

8.2. Example

Consider the following FSIDE [72-76]

$$D^{\alpha}f(t) = \frac{7}{12}t^4 - \frac{5}{6}t^3 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^t (s+t)f(s)\,ds + \int_0^t s f(s)\,dB(s)$$

, $t \in [0,1]$

with $f(0) = 0$. This FSIDE does not have analytical solution. The numerical results for $f(t)$ by our presented method for $\alpha = 0.25, 0.5, 0.75$ with $m = 8$ are plotted in (Figure 3). Moreover, in order to compare our method with the method of P panels M-point Newtoncotes, you can see [72]. In [72], Taheri at al obtained the results with $m = 15$ while we obtained our result in (Figure 3) by Tau method with $m = 8$. These results demonstrated the validity and high accuracy of our presented method.

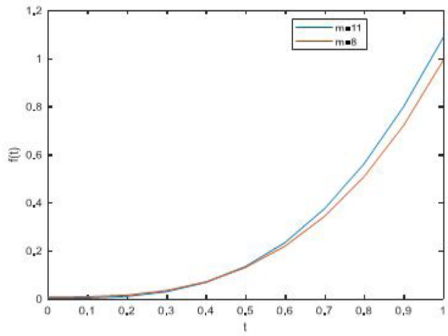

Figure 2: The graph of the approximate solution of example 1 by Tau method for $\sigma = 1$, $\alpha = 0.75$ with $m = 8,11$.

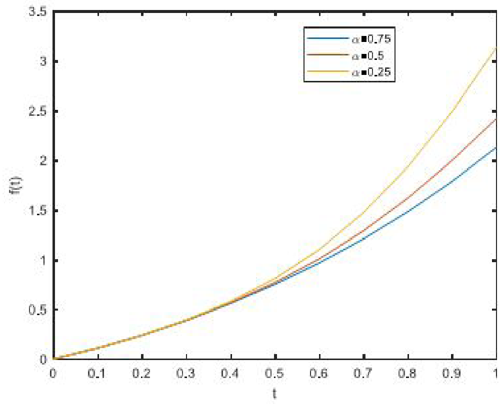

Figure 3: The graph of the approximate solution of example 2 by the Tau method for $\alpha = 0.25, 0.5, 0.75$ with $m = 8$.

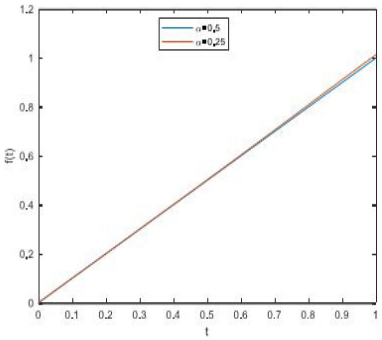

Figure 4: The graph of the approximate solution of example 3 for $\alpha = 0.25$, 0.5 with m=10.

8.3. Example

Let us consider the following FSIDE [72]

$$D^{\alpha} f(t) = \frac{\Gamma(2)t^{1-\alpha}}{\Gamma(3-\alpha)} + \frac{t^3}{3} + \int_0^t sf(s)ds + \int_0^t f(s)\,dB(s) ,$$

$t \in [0,1]$

with $f(0) = 0$.

For solving the example 6.3 we apply the presented method and take $k2(t,s) = 1$. This problem has been solved for $\alpha = 0.5$ with 10 m= and the results are shown in (Figure 4). Moreover, you can see the solution of this equation by the method of P Panels M-point Newton-cotes in [72].

9. Conclusion

In this paper, we presented a technique based on the operational matrices of the Tau method to approximate numerical solutions of fractional stochastic integro-differential equations. This method reduced the FSIDEs into a system of linear algebraic equations. The numerical results demonstrated that this approach solves the problems of fractional stochastic integro-differential equations effectively.

Referecnces

1. Kloeden LL, Pearson RA. The numerical solution of stochastic differential equations. J. Aust. Math. Soc. Ser. B. Appl. Math. 1992; 20: 8-12.

2. Helfand E. Numerical Integration of Stochastic Differential Equations. Bell Syst. Tech. J. 1979; 58: 2289-99.

3. Greiner A, Strittmatter W, Honerkamp J. Numerical integration of stochastic differential equations. J. Stat. Phys. 1988; 51: 95-108.

4. Golec J, Ladde G. Euler-type approximation for systems of stochastic differential equations. J. Appl. Math. Simul. 1989; 2: 239-49.

5. Kanagawa S. The Rate of Convergence for Approximate Solutions of Stochastic Differential Equations. Tokyo J. Math. 1989; 12: 33-48.

6. Kloeden PE, Platen E. A survey of numerical methods for stochastic differential equations. Stoch. Hydrol. Hydraul. 1989; 3: 155-78.

7. Bally V. Approximation for the solutions of stochastic differential equations. i: l^p-convergence. Stochastics Stoch. Reports. 1989; 28: 209-46.

8. Bally V. Approximation for the solutions of stochastic differential equations: ii strong convergence. Stochastics Stoch. Reports. 1989; 28: 357-85.

9. Bally V. Approximation for the solutions of stochastic differential equations. III: Jointly weak convergence. Stochastics and Stochastics Reports. 1990; 30: 171-91.

10. Kurtz TG, Protter P. Weak limit theorems for stochastic integrals and stochastic differential equations. Ann. Probab. 19911; 9: 1035-70.

11. Kloeden PE, Platen E, Schurz H. The Numerical Solution of Nonlinear Stochastic Dynamical Systems: a brief introduction. Int. J. Bifurc. Chaos.1991; 1: 277-86.

12. Kloeden PE, Platen E. Stratonovich and Ito Stochastic Taylor Expansions. Math. Nachrichten. 1991; 151: 33-50.

13. Gorostiza LG. Rate of convergence of an approximate solution of stochastic differential equations. Stochastics. 1980; 3: 267-76.

14. Kloeden P E, Platen E. Higher-order implicit strong numerical schemes for stochastic differential equations. J. Stat. Phys. 1992; 66: 283-314.

15. Bensoussan A, Glowinski R, Raşcanu A. Approximation of some stochastic differential equations by the splitting up method. Appl. Math. Optim. 1992; 25: 81-106.

16. Goodlett ST, Allen EJ. A variance reduction technique for use with the extrapolated Euler method for numerical solution of stochastic differential equations. Stoch. Anal. Appl.1994; 12: 131-40.

17. Komori Y, Saito Y, Mitsui T. Some issues in discrete approximate solution for stochastic differential equations. Comput. Math. Appl.1994; 28: 269-78.

18. Hofmann N, Platen E. Stability of weak numerical schemes for stochastic differential equations. Comput. Math. Appl. 1994; 28: 45-57.

19. Gelbrich M.Simultaneous time and chance discretization for stochastic differential equations. J. Comput. Appl. Math. 1995; 58(3): 255-89.

20. Komori Y, Mitsui T. Stahle ROW-Type Weak Scheme for Stochastic Differential Equations. Monte Carlo Methods Appl. 1995; 1: 279-300.

21. Bally V, Talay D. The law of the Euler scheme for stochastic

differential equations. Probab. Theory Relat. Fields. 1996; 104: 43-60.

22. Lépingle D. Euler scheme for reflected stochastic differential equations. Math. Comput. Simul. 1995; 38: 119-26.

23. Kanawaga S. Error Estimations for the Euler-Maruyama Approximate Solutions of Stochastic Differential Equations. Monte Carlo Methods Appl. 1995; 1: 165-71.

24. Greenside HS, Helfand E. Numerical integration of stochastic differential equations. Bell Syst. Techn. J. 1981; 60: 1927-40.

25. Hofmann N. Stability of weak numerical schemes for stochastic differential equations. Math. Comput. Simul. 1995; 38: 63-8.

26. Castell F, Gaines J. An efficient approximation method for stochastic differential equations by means of the exponential Lie series. Math. Comput. Simul. 1995; 38: 13-9.

27. Grecksch W, Kloeden PE. Time-discretisedGalerkin approximations of parabolic stochastic PDEs. Bull. Austral. Math. Soc. 1996; 54: 79-85.

28. Grorud A, Talay D. Approximation of Lyapunov Exponents of Nonlinear Stochastic Differential Equations. SIAM J. Appl. Math. 1996; 56: 627-50.

29. Burrage K, Burrage PM. High strong order explicit Runge–Kutta methods for stochastic ordinary differential equations. Appl. Numer. Math. 1996; 22: 81-101.

30. Gyöngy I, Nuarlart D. Implicit scheme for stochastic partial differential equations driven by space-time white noise. Potential Anal. 1997; 7: 725-57.

31. Komori Y, Mitsui T, Sugiura H. Rooted tree analysis of the order conditions of rowtype scheme for stochastic differential equations. BIT Numer. Math. 1997; 37: 43-66.

32. Burrage K, Burrage PM, Belward JA. A bound on the maximum strong order of stochastic Runge-Kutta methods for stochastic ordinary differential equations. BIT Numer. Math. 1997; 37: 771-80.

33. Denk G, Schäffler S. Adams methods for the efficient solution of stochastic differential equations with additive noise. Computing. 1997; 59: 153-61.

34. Gaines JG, Lyons TJ. Variable Step Size Control in the Numerical Solution of Stochastic Differential Equations. SIAM J. Appl. Math. 1997; 57: 1455-84.

35. Guo SJ. On the mollifier approximation for solutions of stochastic differential equations. J. Math. Kyoto Univ. 1982; 22: 243-54.

36. Kohatsu-Higa A, Ogawa S. Weak rate of convergence for an Euler scheme of nonlinear SDE's. Monte Carlo Methods Appl. 1997; 3: 327-45.

37. Chan KS, Stramer O. Weak consistency of the Euler method for numerically solving stochastic differential equations with discontinuous coefficients. Stoch. Process. Their Appl. 1998; 76: 33-44.

38. Allen EJ, Novosel SJ, Zhang Z. Finite element and difference approximation of some linear stochastic partial differential equations. Stochastics Stoch. Reports. 1998; 64: 117-42.

39. Burrage K, Burrage PM. General order conditions for stochastic Runge-Kutta methods for both commuting and non-commuting stochastic ordinary differential equation systems. ApplNumer. Math. 1998; 28: 161-77.

40. Abukhaled MI, Allen EJ. A recursive integration method for approximate solution of stochastic differential equations. Int. J. Comput. Math. 1998; 66: 53-66.

Platen E. An introduction to numerical methods for stochastic differential equations. ActaNumer. 1999; 8: 197-246.

41. Misawa T. Numerical integration of stochastic differential equations by composition methods. Dyn. Syst. Differ. Geom. 2000; 1180: 166-90.

42. Franklin JN. Difference Methods for Stochastic Ordinary Differential Equations. Math. Comput. 1965; 19: 552-61.

43. Famelis IT, Xanthos F,Papageorgiou G. "Numerical solution of Stochastic Differential Eequations with Additive Noise by Runge-Kutta methods". J. Numer. Anal. Ind. Appl. Math. 2009; 4(3-4): 171-80.

44. Janssen R. Discretization of the Wiener-process in difference-methods for stochastic differential equations. Stoch. Process. Their Appl. 1984; 18: 361-69.

45. Klauder JR, Petersen WP. Numerical Integration of Multiplicative-Noise Stochastic Differential Equations. SIAM J. Numer. Anal. 1985; 22: 1153-66.

46. Haworth DC, Pope SB. A second-order Monte-Carlo method for the solution of the Itô stochastic differential equation. Stoch. Anal. Appl. 1986; 4: 151-86.

47. Drummond IT, Hoch A, Horgan RR. Numerical integration of stochastic differential equations with variable diffusivity. J. Phys. A. Math. Gen. 1986; 19: 3871-81.

48. Chang CC. Numerical solution of stochastic differential equations with constant diffusion coefficients. Math. Comput. 1987; 49: 523.

49. Wilkie J. Numerical methods for stochastic differential equations. Phys. Rev. E - Stat. Physics, Plasmas, Fluids, Relat. Interdiscip. Top. 2004; 70: 1-4.

50. Mao X. Stochastic differential equations and applications. 2nd ed., Ellis Horwood Publishing, Chichester, UK. 2007.

51. Zhou S, Jin H. Strong convergence of implicit numerical methods for nonlinear stochastic functional differential equations. J. Comput. Appl. Math. 2017; 324: 241-57.

52. Li X, Mao X, Yin G. Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment and stability. IMA J. Numer. Anal. 2018; 1: 1-46.

53. Bao J, Böttcher B, Mao X, Yuan C. Convergence rate of numerical solutions to SFDEs with jumps. J. Comput. Appl. Math. 2011; 236: 119-31.

54. Komori Y. Weak second-order stochastic Runge-Kutta methods for non-commutative stochastic differential equations. J. Comput. Appl. Math. 2007; 206: 158-73.

55. Khodabin M, Maleknejad K, Rostami M, Nouri M. Numerical solution of stochastic differential equations by second order Runge-Kutta methods. Math. Comput. Model. 2011; 53: 1910-20.

56. Momani S, Al-smadi MH, FreihatA. Analytical Study of Fractional-Order Multiple Chaotic FitzHugh-Nagumo Neurons Model Using Multistep Generalized Differential Transform Method. Abstr. Appl. Anal. 2014; 10 Pages.

57. Gevorkyan MN, Velieva TR, Korolkova AV, Kulyabov DS, Sevastyanov LA. Stochastic Runge-Kutta software package for stochastic differential equations. Adv. Intell. Syst. Comput. 2016; 470: 169-79.

58. Shukla HS, Tamsir M. Numerical solution of nonlinear sine–Gordon equation by using the modified cubic B-spline differential quadrature method. Beni-Suef Univ. J. Basic Appl. Sci. 2018; 7: 359-66.

59. Ortiz EL. The Tau method. SIAM J. Numer. Anal. 1969; 6:480-92.

60. Liu KM, Pan CK. The Automatic Solution to Systems of Ordinary Differential Equations by the Tau Method. Comput. Math. with Appl. 1999; 38: 197-210.

61. Rahimi-Ardabili MY, Shahmorad S. Iterative numerical solution of non-linear integrodifferential equations by the Tau method. Appl. Math. Comput. 2007; 193: 514-22.

62. SedighiHafshejani J, KarimiVanani S, Esmaily J. Operational Tau Approximation for Neutral Delay Differential Systems. J. Appl. Sci. 2011; 11: 2585-91.

63. Yisa BM, Adeniyi RB. On the generalization of the error and error estimation process of Ortiz's recursive formulation of the tau method. J. Niger. Math. Soc. 2014; 34: 70-82.

64. Koppelaar H, Kordestani-Moghadan P, Khan K, Kouhkani S, Segers G, Warmerdam M. Reaction Time Improvements by Neural Bistability. Behav. Sci. (Basel). 2019; 9: 1-14.

65. Lanczos C. Trigonometric Interpolation of Empirical and Analytical Functions. J. Math. Phys. 1938; 17: 123-99.

66. Ortiz EL, Samara H. An Operational approach to the Tau method for the numerical solution of non-linear differential equations. Computing. 1981; 27: 15-25.

67. Cruz-Santiago R, Hernandez-Aguilar C, Lopez-Bonilla J. An Application of the Tau Method. Comput. Appl. Math. Sci. 2018; 31: 15-7.

68. Khani A, Shahmorad S. A simple numerical method for solving non-linear Volterraintegro-differential equations. Arab. J. Sci. Eng. 2009; 34: 209-19.

69. Gepreel KA, Nofal TA, Alasmari AA. Exact solutions for nonlinear integro-partial differential equations using the generalized Kudryashov method. J. Egypt. Math. Soc. 2017; 25: 438-44.

70. Clements D, Anderson B. Well-behaved Itô equations with simulations that always misbehave. IEEE Trans. Automat. Contr. 1973; 18: 676-7.

71. Taheri Z, Javadi S, Babolian E. Numerical solution of stochastic fractional integrodifferential equation by the spectral collocation method. J. Comput. Appl. Math. 2017; 321: 336-47.

72. Słomiński L. On reflected Stratonovich stochastic differential equations. Stoch. Process. Their Appl. 2015; 125: 759-79.

73. Vanani SK, Aminataei A. Operational Tau approximation for a general class of fractional integro-differential equations. Comput. Appl. Math. 2011; 30: 655-74.

74. Kouhkani S, Koppelaar H, Abri M. Numerical solution of fractional neutral functionaldifferential equations by the operational Tau method. Casp. J. Comput. Math. Eng. 2017; 5-19.

75. Mirzaee F, Samadyar N. On the numerical solution of fractional stochastic integrodifferential equations via meshless discrete collocation method based on radial basis functions. Eng. Anal. Bound. Elem. 2018; 100: 246-55.

76. Onumanyi P, Ortiz EL. Numerical solution of stiff and singularly perturbed boundary value problems with a segmented-adaptive formulation of the Tau method. Mathematics of Computation. 1984; 43: 189-203.

77. Khajeh HG, Ortiz EL. Upper and lower estimation for the Tau method and related polynomial techniques. 1991; 22: 81-7.

78. Yousefi A, Babolian E, Javadi S. The convergence of operational Tau method for solving a class of nonlinear Fredholm fractional integro-differential equations on Legendre basis. 2017.

79. Heydari MH, Avazzadeh Z, Mahmoudi Z. Chebyshev cardinal wavelaats for nonlinear stochastic differential equations driven with variable-order fractional Brownian motion. Chaos, Solitons and Fractals. 2019; 124: 102-24.