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Measures of Inequality in Vectors Distributions

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1. Abstract

In this paper, first, we propose two measures of inequality in multivariate distributions. One of these measures generalizes the Gini index to the multidimensional case. We study some properties of these two measures. We then introduce the notion of intrinsic inequality and inequality angle. Contrarily to the two defined inequality indexes, the intrinsic inequality will have the particularity to be similitude invariant. We give a geometric interpretation for it. Finally, we do some empirical applications of these different defined measures.

2. Introduction

Measuring inequality is a subject widely studied. Several measures exist. One of the most known is the Gini index developed by Gini (1921) to measure the inequality income. This index of inequality is only applied to univariate and non-negative distributions. Different researches have been done to extend the Gini index to negative variables: for instance, [7] propose an approach of extension of the Gini Index to negative income. The generalization of the inequality measure from univariate distributions to multivariate distributions has also deeply investigated by different researchers. [5] proposes a [6] index up to a constant and investigates on which constant guarantees the index be bounded by one. [8] introduced the notion of the volume of a convex body associated to a distribution to extend the notion of Lorenz curve. We can mention as well the works of [9-11] proposed a multivariate Gini mean difference. The weakness of these different extensions of the Gini indexes is the fact that they can possibly be equal to zero even for non-constant variables [12] introduced a multivariate Gini index, associated with a concentration surface. [1] propose two multivariate indices that generalize the Gini index; these in Decancq troduced measures inherit the main properties of the Gini index. They applied their measure to the Iris data to corroborate the idea that the iris versicolor is the hybrid of the iris setosa and the iris virginica. [4] extend to the multidimensional attributes case the axioms used to characterize the generalized Gini social evaluation orderings for one- dimensional distributions. BARRY C. ARNOLD (2005) proposes the volume of the Lorenz zonoid as a strong candidate for the title of natural extensions of the Gini index to higher dimensions. We can also mention [3] for their measure of well-being with a multidimensional Gini index. [2] constructed a multivariate Gini index satisfying simultaneously the condition of Correlation Increasing Majorization and the condition of Weighting of Attributes under Unidirectional Co-monotonicity. In this paper, we introduce the notion of inequality measure of a distribution with respect to a reference point by considering the ratio of the mean distance in the distribution to the mean distance of the distribution to the reference point. We then deduce two relative measures of inequality $\mathcal{G}$ and $\mathfrak{T}$ by choosing as reference the worst point of the distribution. This approach seems to non-suggested yet in the literature and the different extensions in multiple dimensions are generally for multivariate non negative distributions. The index $\mathcal{G}$ extends the Gini index to the multivariate distributions with possibly negative coordinates and the second one $\mathfrak{T}$ is very similar to the Pietra in the case the distribution is univariate. We study the properties of these two measures. They inherit the continuity property, invariance by symmetry with respect to the coordinates. And instead of the vector scale invariance, they have a real scale invariance and mainly the continuity of the vector scaling application φX and ρX for all

$X = (X_1, \dots, X_q) \in \mathbb{R}^q$ where:

$$\varphi X = \begin{array}{ccc} \mathbb{R}_+^q & \rightarrow & [0,1] \\ (\beta_1, \dots, \beta_q) & \longmapsto & \mathcal{G}\left((\beta_1 X_1, \dots, \beta_q X_q)\right) \end{array}$$

and

$$\rho X = \begin{array}{ccc} \mathbb{R}_+^q & \rightarrow & [0,1] \\ (\beta_1, \dots, \beta_q) & \longmapsto & \mathfrak{I}\left((\beta_1 X_1, \dots, \beta_q X_q)\right) \end{array}$$

Moreover, the application ρX is continuous and monotone in all directions. The vector scaling changes the indexes if the coordinates vectors are non collinear. More, precisely, the index $\mathfrak{I}$ gets closer to in the inequality of the more weighted coordinate when we do a vector scaling. That is to say: for all X and Y two vector with possibly unequal length, such that $\mathfrak{I}(X) \leq \mathfrak{I}(Y)$ then for Z = (X, Y), for all α > 0 and β > 0, $\mathfrak{I}$((αX, βY)) ≤ $\mathfrak{I}$(Z) if and only if α ≥ β. For instance if X has less inequality than Y then $\mathfrak{I}$(2X, Y) ≤ $\mathfrak{I}$(X, Y) because in (X, Y) is less weighted than in (2X, Y).

We also introduce the notion of intrinsic inequality and inequality angle. We give a geometric illustration of these two notions. The intrinsic inequality index is defined to be similitude-invariant i.e., invariant by any application that multiplies distances by a constant. We finally do some little applications. With the iris data, considering the indexes G and I index we propose, we prove as G. A. Koshevoy And K. Mosler (1997) that iris versicolor is an hybrid of iris setosa and iris virginica.

3. Notations and Definitions

We consider the problem of constructing an index of inequality in the distribution of random vectors. We need to adopt some notations and definitions.

Let q ∈ N*. Let ‖.‖ be the Euclidean norm defined on the vectors space $\mathbb{R}^q$.

$$\forall x = (x_1, \dots, x_q)' \in \mathbb{R}^q, \|X\| = \left(\sum_{i=1}^{q} |x_i|^2\right)^{\frac{1}{2}}$$

We denote by $\mathcal{L}^0(\mathbb{R}^q)$ the set of random vectors taking values in $\mathbb{R}^q$ and we define the following subsets of $\mathcal{L}^0(\mathbb{R}^q)$:

$$\mathcal{L}^1(\mathbb{R}^q) = \{X \in \mathcal{L}^0(\mathbb{R}^q) | \mathbb{E}[\|X\|] < +\infty\}$$

and

$$\mathcal{L}^2(\mathbb{R}^q) = \{X \in \mathcal{L}^0(\mathbb{R}^q) | \mathbb{E}[\|X\|^2] < +\infty\}$$

Let $\mathcal{M}^0(\mathbb{R}^q)$ be the set of distributions on $\mathbb{R}^q$. We then define:

$$\mathcal{M}^1(\mathbb{R}^q) = \left\{F \in \mathcal{M}^0(\mathbb{R}^q) | \int_{\mathbb{R}^q} \|X\| dF(x) < +\infty\right\}$$

and

$$\mathcal{M}^2(\mathbb{R}^q) = \left\{F \in \mathcal{M}^0(\mathbb{R}^q) | \int_{\mathbb{R}^q} \|X\| dF(x) < +\infty\right\}$$

In other words, $\mathcal{M}^1(\mathbb{R}^q)$ is the set of the distributions of the random vectors in $\mathcal{L}^1(\mathbb{R}^q)$. For any vector X in $\mathcal{L}^0(\mathbb{R}^k)$, we will denote by $F_X \in \mathcal{M}^0(\mathbb{R}^k)$ its distribution function.

Let $\mathcal{L}^0$ be the set of all random vectors with finite expectation of norm and $\mathcal{M}^1$ the set of their distributions.

$$\mathcal{L}^1 = \bigcup_{q=1}^{\infty} \mathcal{L}^1(\mathbb{R}^q) \text{ and } \mathcal{M}^1 = \bigcup_{q=1}^{\infty} \mathcal{M}^1(\mathbb{R}^q)$$

We denote by E the following set:

$$\mathcal{E} = \{I | I: \mathcal{M}^1 \to [0,1]\}$$

For all random vector X, with distribution F, the notation I(X) will also design I(F). So I(X) = I(Y) = I(F) if the vectors X and Y have the same distribution F .

Definition 3.1. Let I ∈ E. We define a total relation of order $\preceq^I$ on $\mathcal{M}^1$ and $\mathcal{L}^1$ as follows:

$$\forall (F,G) \in \mathcal{M}^1 \times \mathcal{M}^1, F \preceq^I G \Leftrightarrow I(F) \geq I(G)$$

$$F \prec^I G \Leftrightarrow I(F) \geq I(G)$$

and

$$\forall (X,Y) \in \mathcal{L}^1 \times \mathcal{L}^1, X \preceq^I Y \Leftrightarrow I(X) \geq I(Y)$$

$$X \prec^I Y \Leftrightarrow I(X) \geq I(Y)$$

We define a partial relation of order ≤ between vectors in a same space $\mathbb{R}^q$.

Definition 3.2. Let $q \in \mathbb{N}^*$.

$$\forall x = (x_1, \dots, x_q)' \epsilon \mathbb{R}^q, \forall y' = (y_1, \dots, y_q) \epsilon \mathbb{R}^q, x \leq y \text{ iff } (x_1 \leq y_1, \dots, x_q \leq y_q)$$

and

$$\forall x = (x_1, \dots, x_q)' \epsilon \mathbb{R}^q, \forall y' = (y_1, \dots, y_q) \epsilon \mathbb{R}^q, x \leq y, and\ x \neq y$$

For instance, (1, -3, 7, 0, 5)’ ≤ (2, 0, 7, 3, 5)’. But (4, 7, 0, -5, 1)’ and (0, 8, -1, 6, 2)’are not in relation.

We can now define two relative measures of inequality in vectors distribution.

Definition 3.3. Let $F \in \mathcal{M}^1 \mathbb{R}^q$. Let $a \in \mathbb{R}^q$ be constant vector. Two relative inequality indexes of F with respect to a are defined as follows:

$$\mathcal{G}_\alpha(F) = \frac{\mathbb{E}[\|X - Y\|]}{2. \mathbb{E}[\|X - \alpha\|]}$$

and

$$\mathfrak{T}_a(F) = \sqrt{\frac{\mathbb{E}[\|X - \mathbb{E}(X)\|^2]}{\mathbb{E}[\|X - a\|^2]}}$$

where X and Y are two independent random vectors in $\mathcal{L}^1 \mathbb{R}^q$ and with distribution F.

We denote:

$$D(F) = \mathbb{E}[\|X - Y\|],$$

and

$$\Delta(F) = \sqrt{\mathbb{E}[\|X - \mathbb{E}(X)\|^2]},$$

where X and Y have the same distribution F and are independent. We can interpret D(F) as the mean difference in the distribution F. And $\Delta(F)$ can be interpreted as the mean distance of the distribution to the ideal distribution

that is the constant $\mathbb{E}(X)$. The term $\mathbb{E}[\|X - a\|]$is the mean distance of the distribution F to the reference distribution that is the constant a.

Let's now focus on the different properties of these two relative inequality indexes.

For recall: for any distribution $F \in \mathcal{M}^2(\mathbb{R}^q)$, and $X \in \mathcal{M}^1(\mathbb{R}^q)$ following the distribution F, the notations D(X), Δ (X), $\mathcal{G}$(X) and $\mathfrak{T}$ (X) are without any ambiguity respectively equal to D(F), Δ (X), $\mathcal{G}$(X) and $\mathfrak{T}$ (X).

Proposition 1. Let $F \in \mathcal{M}^2(\mathbb{R}^q)$ and $X \in \mathcal{L}^1(\mathbb{R}^q)$ following the distribution F.

We have:

1. $0 \leq D(F), 0 \leq \Delta(F)$
2. $D(F) = \Delta(F) = 0 \Leftrightarrow \exists c \in \mathbb{R}^q | \mathbb{P}(X = c) - 1$
3. $\forall b \in \mathbb{R}^q, D(X + b) = D(F)$ and $\Delta(X + b) = \Delta(X)$

4. $\forall \Pi$ permutation matrix, $D(\Pi X) = D(X)$ and $\Delta(\Pi X)$
5. $\forall P$ orthogonal matrix, $D(PX) = D(X) and\ \Delta(PX) = \Delta(X)$
6. D andΔ are continuous w.r.t weak convergence of distribution

Proof. Let $F \in \mathcal{M}^2(\mathbb{R}^q)$. Let $X, Y \in \mathcal{L}^2(\mathbb{R}^q)$ independent and following the distribution F.

1. $D(X) = \mathbb{E}[\|X - Y\|] > 0$ and $\Delta(X) = \sqrt{\mathbb{E}[\|X - \mathbb{E}(X)\|^2]}$
2. $D(X) = 0, X \perp Y) \Leftrightarrow (X - Y\ a.s\ X \perp Y) \Rightarrow X$ constant a.s When X is constant, it is obvious that D(X) = 0. We do a similar reasoning for Δ.
3. Let $b \in \mathbb{R}^q$

$$D(X + b) = \|(X + b) - (Y + b)\| = \|X - Y\| = D(X)$$

Δ is also invariant by translation.

4. Note that any permutation matrix Π is an orthogonal matrix.

$$\forall P \text{ orthogonal matrix, } P'P = I_q$$

so for all P orthogonal matrix, we have:

$$\|PX - PY\| = \|P(X - Y)\| = \|(X - Y)'P'P(X - Y)\| = \|X - Y\|$$

5. Let $\mu = \mathbb{E}[X]$ be. Let's consider the applications:

$$\begin{array}{llll} \varphi_1: & \mathbb{R}^q \times \mathbb{R}^q & \to & \mathbb{R}_+ \\ & (x, y) & \mapsto & \dfrac{\mathbb{E}[\|x - y\|]}{2\mathbb{E}[\|x - a\|]} = \dfrac{[\|x - y\|]}{2[\|x - a\|]} \end{array}$$

and

$$\begin{array}{llll} \varphi_2: & \mathbb{R}^q \times \mathbb{R}^q & \to & \mathbb{R}_+ \\ & (x, y) & \mapsto & \sqrt{\dfrac{\mathbb{E}[\|x - y\|^2]}{2\mathbb{E}[\|x - a\|^2]}} = \sqrt{\dfrac{[\|x - y\|^2]}{2[\|x - a\|^2]}} \end{array}$$

The applications φ_1 and φ_2 are continuous at any point different from a.

Let $((X_n, Y_n))_n$ a sequence of random vectors lying on $\mathbb{R}^q \times \mathbb{R}^q$ such that for all n, Xn and Yn are independent and follow the same distribution Fn; we also assume $(F_n)_n$ converges to F. Using the mapping theorem, we can write:

$$\mathcal{G}_a(F_n) = \varphi_1(X_n, Y_n) \underset{\rightarrow}{\mathcal{L}} \varphi_1(X, Y) = \mathcal{G}_a(F)$$

and

$$4\mathfrak{T}_a(F_n) = \varphi_2(X_n, Y_n) \underset{\rightarrow}{\mathcal{L}} \varphi_2(X, Y) = \mathfrak{T}_a(F)$$

Proposition 2. Let $F \in \mathcal{M}^2(\mathbb{R}^q)$. Let $X \in \mathcal{L}^2(\mathbb{R}^q)$ be a random vector following the distribution FX = F and let $a \in \mathbb{R}^q$. We have:

1. $0 \leq \mathcal{G}_a(F), 0 \leq \mathfrak{T}_a(F)$
2. $\mathcal{G}_a = \mathfrak{T}_a = 0 \Leftrightarrow \exists c \in \mathbb{R}^q | \mathbb{P}(X = c) = 1$
3. If $\mathbb{P}(a \leq X) = 1$, then $\forall b > 0_{\mathbb{R}^q}$,

$$\mathcal{G}_a(X + b) < \mathcal{G}_a(X) \text{ and } \mathfrak{T}_a(X + b) < \mathfrak{T}_a(X)$$

4. $\forall \beta \in \mathbb{R}^*, \mathcal{G}_{\beta a}(\beta X) = \mathcal{G}_a(X)$ and $\mathfrak{T}_{\beta a}(\beta X) = \mathfrak{T}_a(X)$
5. $\forall \Pi$ permutation matrix,

$$\mathcal{G}_{\Pi a}(\Pi X) = \mathcal{G}_a(X) \text{ and } \mathfrak{T}_{\Pi a}(\Pi X) = \mathfrak{T}_a(X)$$

6. $\forall P$ orthogonal matrix,

$$\mathcal{G}_{Pa}(PX) = \mathcal{G}_a(X) \text{ and } \mathfrak{T}_{Pa}(PX) = \mathfrak{T}_a(X)$$

7. $\mathcal{G}_a$ and $\mathfrak{T}_a$ are continuous w.r.t weak convergence of distributions.

Proof. Let $F \in \mathcal{M}^2(\mathbb{R}^q)$. Let $X, Y \in \mathcal{L}^2(\mathbb{R}^q)$ be two independent random vectors following the distribution $F_X = F$ and let $a \in \mathbb{R}^q$.

1. $\mathcal{G}_a$ and $\mathfrak{I}_a$ are the ratio of two non-negative terms. They are obviously positive.
2. $\mathcal{G}_a = \mathfrak{I}_a = 0 \Leftrightarrow D = \Delta = 0$
3. Let's assume $\mathbb{P}(a \leq X) = 1$. Let $b \in \mathbb{R}^q$ such that $b > 0_{\mathbb{R}^q}$.

$$\forall i \in \{1, \dots, q\}, 1 = \mathbb{P}(a \leq X) \leq \mathbb{P}(a_i \leq X_i) \Rightarrow \forall i \in \{1, \dots, q\}, \mathbb{P}(X_i - a_i \geq 0) = 1$$

$$b > 0_{\mathbb{R}^q} \Rightarrow \forall i \in \{1, \dots, q\} b_i \geq 0 \geq, \exists j \in \{1, \dots, q\} | b_j > 0$$

So

$$\forall i \in \{1, \dots, q\} X_i - a_i + b_i \geq X_i - a_i \geq 0 \; a.s \; and \; \exists j \in \{1, \dots, q\} | X_i - a_i + b_j > X_j - a_j \geq 0 a.s$$

$$\begin{aligned}\mathcal{G}_a(X+b) &= \frac{D(X+b)}{2\mathbb{E}[\|X+b-a\|]} \\ &= \frac{D(X)}{2\mathbb{E}\left[\sqrt{\sum_{i=1, i\neq j}^{q}(X_i - a_i + b_i)^2 + (X_j - a_j + b_j)^2}\right]} \\ &< \frac{D(X)}{2\mathbb{E}\left[\sqrt{\sum_{i=1, i\neq j}^{q}(X_i - a_i)^2 + (X_j - a_j)^2}\right]} \\ &< \frac{D(X+b)}{2\mathbb{E}[\|X-a\|]}\end{aligned}$$

The same reasoning is convenient for Δ.

4. Let $\beta \in \mathbb{R}^*$.

$$\mathcal{G}_{\beta a}\,(\beta X) = \frac{\mathbb{E}[\|\beta X - \beta Y\|]}{2\mathbb{E}[\|\beta X - \beta a\|]} = \frac{|\beta|\mathbb{E}[\|X-Y\|]}{2|\beta|\mathbb{E}[\|X-a\|]} = \frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|X-a\|]} = \mathcal{G}_a\,(X)$$

5. Permutation matrices are orthogonal matrices and orthogonal matrices

conserve norms.

From the first proposition, we remark that D(F) and Δ (F) are translation invariant. The second one shows that G0 and I0 are real scale invariant contrary to the multivariate Gini index of Koshevoy and Mosler (1997) which is vector scale invariant. The real scale invariance seems us to be more realistic than the vector scale invariance for the simple reason that real scales conserve the relative importance of the different axes of dimensions of the random vector whereas the vector scales modify that; in fact one axis can become more weighted than another one which initially was more weighted and such transformation will completely change the inequality in the distribution. For example: consider a random vector $Z = (X, Y) \in \mathbb{R}^+ \times \mathbb{R}^+ \; and \; b = (u, v) \in \mathbb{R}^* \times \mathbb{R}^*$ such that $\frac{u}{v} \simeq 0$. The inequality in the distribution of $(u.X, v.Y) = v.(\frac{u}{v}.X, Y)$ will almost be equal to the inequality in Y because the axis Y have infinitely been weighted than the axis X so the axis X has lost in importance and this will change the inequality in the distribution. It is as if the projection of Z on any vector subspace conserves its inequality, and this is clearly not realistic. But the product of Z by a real scale does not change the inequality in its distribution.

The third point of the second proposition means that any positive translation of the distribution reduces the inequality when the reference point is almost surely never taken by the random vector. In the case the support space is included in ${\mathbb{R}_+}^q$, we can simply take as reference point $0_{\mathbb{R}^q}$.

Theorem 3.1. Let $F \in (\mathcal{M}^2(\mathbb{R}^q))$. Let $X = \in \mathcal{L}^2(\mathbb{R}^q)$ be a random vector following the distribution F.

$$\forall a \in \mathbb{R}^q, \text{ such that } \mathbb{P}(X = a) < 1, \text{ we have:}$$

$$0 \leq \mathcal{G}_a\,(F) < 1 \; and \; 0 \leq \mathfrak{I}_a(F) < 1$$

Furthermore,

$$\mathcal{G}_a\,(F) = \mathfrak{I}_a(F) = 0 \Leftrightarrow F \text{ is a one-point distribution}$$

Proof. Let $F \in (\mathcal{M}^2(\mathbb{R}^q))$. Let $X =\in \mathcal{L}^2(\mathbb{R}^q)$ be two independent random vectors following the distribution F.

1.

$$\mathcal{G}_a(F) = \frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|X-a\|]} \leq \frac{\mathbb{E}[\|X-a\|] + \mathbb{E}[\|y-a\|]}{2\mathbb{E}[\|X-a\|]} \leq \frac{\mathbb{E}[\|X-a\| + \|X-a\|]}{2\mathbb{E}[\|X-Y\|]} \leq 1$$

Furthermore

$$\mathbb{E}[\|X-Y\|] = 1 \Rightarrow \mathbb{E}[\|X-Y\|] = \mathbb{E}[\|X-a\| + \|X-a\|] \Rightarrow \exists \curlywedge \mathbb{R} | \forall \omega - a = \curlywedge (Y(\omega) - a)$$

so

$$(\mathbb{E}[\|X-a\|] = \mathbb{E}[\|Y-a\|], \mathbb{E}[\|X\|] = \mathbb{E}[\|Y\|]) \Rightarrow \curlywedge = 1 \Rightarrow X = Y = cste\ a.s$$

As X≠a almost surely, we have:

$$X = Y = cste\ a.s \Rightarrow \mathcal{G}_a(F) = 0$$

So $\mathcal{G}_a(F) < 1$.

We also note that:

$$\mathcal{G}_a(F) = 0 \Leftrightarrow X = Y a.s \Leftrightarrow F \text{ is a one-point distribution}$$

2.

$$\begin{aligned}
\mathfrak{T}_a(F)^2 &= \frac{\mathbb{E}[\|X-\mathbb{E}[X]\|^2]}{\mathbb{E}[\|X-a\|^2]} \\
&= \frac{\mathbb{E}[\|X-\mathbb{E}[X]\|^2]}{\mathbb{E}[\|(X-\mathbb{E}[X]) - (a-\mathbb{E}[X])\|^2]} \\
&= \frac{\mathbb{E}[\|X-\mathbb{E}[X]\|^2]}{\mathbb{E}\left[\left\|X-\mathbb{E}[X]\|^2 - 2\mathbb{E}[(X-\mathbb{E}[X])' - (a-\mathbb{E}[X])] + \mathbb{E}\|a-\mathbb{E}[X]\|\right\|^2\right]} \\
&= \frac{\mathbb{E}[\|X-\mathbb{E}[X]\|^2]}{\mathbb{E}[\|X-\mathbb{E}[X]\|^2 - \mathbb{E}[\|a-\mathbb{E}[X]\|^2]} \\
&\leq 1
\end{aligned}$$

As X 6= a almost surely, $\mathbb{E}[\|X-\mathbb{E}[X]\|^2] > 0$ and then $\mathfrak{T}_a(F)^2 < 1$.

Furthermore,

$$\mathfrak{T}_a(F) = 0 \Leftrightarrow \mathbb{E}[\|X-\mathbb{E}[X]\|^2] = 0 \Leftrightarrow X = \text{ constant almost surely.}$$

We can now propose two standard measures of inequality. For that we will define one intermediary notion.

Definition 3.4. Let $F \in (\mathcal{M}^0(\mathbb{R}^q))$. Let $X =\in \mathcal{L}^0(\mathbb{R}^q)$ such that Fx=F. We define:

$$m(F) = (m_1(F), \ldots, m_q(F)) \in \overline{\mathbb{R}^q}$$

Such that:

$$\forall i \in \{1, \ldots, q\}, m_i(F) = \min(0, int\{t \in \mathbb{R} | \mathbb{P}(X_i \leq t) > 0\})$$

When $m(F) \in \mathbb{R}^q$, this will be called the lower bound of the support space of the distribution F or of X.

In the case the support space of the distribution has a lower bound, we will simply take as the reference point this bound. This is in fact the worst value taken by the random vector. The inequality indexes G measures the ratio between the mean distance in the distribution of the vector and the mean distance of the vector to its lower bound. The lower bound of the random can seen as the worst possibility of distribution of the vector where all the distribution is accumulated at this point all most surely, i.e the set of other taken values different from this bound has a measure of probability equal to 0.

The inequality index I is the ratio between the mean distance of the vector to its ideal position which is its mean value and its mean distance to the worst position. We give the following formal definitions.

Definition 3.5. Let $F \in (\mathcal{M}^2(\mathbb{R}^q))$. Such that $m(F) \in \mathbb{R}^q$. Two inequality indexes of F are defined as follows:

$$\mathcal{G}(F) = \frac{\mathbb{E}[\|X - Y\|]}{2.\mathbb{E}[\|X - m(F)\|]}$$

and

$$\mathfrak{T}(F) = \sqrt{\frac{\mathbb{E}[\|X - \mathbb{E}(X)\|^2]}{\mathbb{E}[\|X - m(F)\|^2]}}$$

where X and Y are two independent random vectors in $\mathcal{L}^1(\mathbb{R}^q)$ following the distribution F.

Proposition 3. Let $F \in \mathcal{M}^2(\mathbb{R}^q)$ such that $m(F) \in \mathbb{R}^q$. Let $X \in \mathcal{L}^2(\mathbb{R}^q)$ following the distribution F, and let $b \in \mathbb{R}^q$. We have:

1. $0 \leq \mathcal{G}(F) < 1 \ and\ 0 \leq \mathfrak{T}(F) < 1$
2. $\mathcal{G} = \mathfrak{T} = 0$ iff F is a one-point distribution
3. $\mathcal{G}((X', 0)') = \mathcal{G}(X) \ and\ \mathfrak{T}(X', 0)') = \mathfrak{T}(X)$
4. $\mathcal{G}((X', X')') = \mathcal{G}(X) \ and\ \mathfrak{T}((X', X')') = \mathfrak{T}\,(X)$
5. $\mathcal{G}(X + b) < \mathcal{G}(X) \ iff\ 0_{\mathbb{R}^q} < b + m(F) \ otherwise\ \mathcal{G}(X + b) = \mathcal{G}(X)$
6. $\mathfrak{T}(X + b) < T(F) \ iff\ 0_{\mathbb{R}^q} < b + m(F) \ otherwise\ \mathfrak{T}(X + b) = \mathfrak{T}(F)$
7. $\forall \Pi$ permutation matrix,
$$\mathcal{G}(\Pi \mathrm{X}) = \mathcal{G}(X) \ and\ \mathfrak{T}(\Pi \mathrm{X}) = \mathfrak{T}(X)$$
8. $\forall \beta > 0$,
$$\mathcal{G}(\mathrm{X}) = \mathcal{G}(X) \ and\ \mathfrak{T}(\beta \mathrm{X}) = \mathfrak{T}(X)$$
9. $\mathcal{G}\ and\ \mathfrak{T}$ are continuous w.r.t weak convergence of distributions.
10. The application:
$$\begin{array}{llll} \varphi: & \mathbb{R}_+^q & \longrightarrow & [0,1] \\ & (\beta_1, \dots, \beta_q) & \longmapsto & \mathcal{G}\left((\beta_1 X_1, \dots, \beta_q X_q)'\right) \end{array}$$

is continuous and for all $i \in \{1, \dots, q\}$, the application

$$\begin{array}{llll} \varphi_X^i: & \mathbb{R}_+ & \longrightarrow & [0,1] \\ & \beta & \longmapsto & \mathcal{G}\left((X_1, \dots, X_{i-1}, \beta X_i, X_{i+1} \dots, X_q)'\right) \end{array}$$

is continuous. More generally, for all $U \in \mathcal{L}^1(\mathbb{R}^m), V \in \mathcal{L}^1(\mathbb{R}^n)$, the application $\beta \longmapsto \mathcal{G}((\beta U', V')')$ defined on $\mathbb{R}_+$ is continuous. Furthermore,

$$\varphi_X^i(0) = \ \mathcal{G}(X_{i-1}) \ and\ \lim_{\beta \to +\infty} \mathcal{G}(X_i)$$

11. The application:
$$\begin{array}{llll} \rho X: & \mathbb{R}_+^q & \longrightarrow & [0,1] \\ & (\beta_1, \dots, \beta_q) & \longmapsto & \mathfrak{T}\left((\beta_1 X_1, \dots, \beta_q X_q)'\right) \end{array}$$

is continuous and all its partial applications are continuous and monotone i.e, for all $i \in \{1, \dots, q\}$, the application

$$\begin{array}{llll} \rho_X^i: & \mathbb{R}_+ & \longrightarrow & [0,1] \\ & \beta & \longmapsto & \mathfrak{T}\left((X_1, \dots, X_{i-1}, \beta X_i, X_{i+1} \dots, X_q)'\right) \end{array}$$

More generally, for all $U \in \mathcal{L}^1(\mathbb{R}^m), V \in \mathcal{L}^1(\mathbb{R}^n)$ the application $\beta \longmapsto \mathcal{G}((\beta U', V')')$ defined on $\mathbb{R}_+$ is continuous and monotone. Furthermore,

$$\wedge_{i=1}^q \mathfrak{T}(X_i) \leq \mathfrak{T}\left((\beta_1 X_1, \dots, \beta_q X_q)'\right) \leq \vee_{i=1}^q \mathfrak{T}(X_i)$$

and

$$\mathfrak{T}(X_i) \wedge \mathfrak{T}(X_{-i}) \leq \mathfrak{T}(X_i) \vee \mathfrak{T}(X_{-i})$$

12. $I((X_1, \dots, X_q)')$ is vector scale invariant if and only if $\mathcal{G}(X_i) = \mathcal{G}(X_j)$ for all $i, j \in \{1, \dots, q\}$.

Proof. Let $F \in \mathcal{M}^2(\mathbb{R}^q)$ such that $m(F) \in \mathbb{R}^q$. Let $X \in \mathcal{L}^2(\mathbb{R}^q)$ following the distribution F, and let $b = (b_1, \dots, b_q \in \mathbb{R}^q$.

1. For the two first points, refer to the theorem 3.1.
2.

$$\mathcal{G}((X', o)') = \frac{\mathbb{E}[\|(X',0)' - (Y',0)'\|]}{2\mathbb{E}[\|(X',0)' - m(X',0)'\|]} = \frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|(X',0)' - (m(F)',0)'\|]} = \frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|X-m(F)\|]}$$

We do same for $\mathfrak{T}$.

3.

$$\mathcal{G}((X', X')') = \frac{\mathbb{E}[\|(X',X')' - (Y',Y')'\|]}{2\mathbb{E}[\|(X',X')' - m(X',X')'\|]} = \frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|(X',X')' - (m(F)',m(F)')'\|]}$$

so

$$\mathcal{G}((X', X')') = \frac{\sqrt{2}\mathbb{E}[\|X-Y\|]}{2\sqrt{2}\mathbb{E}[\|X-m(F)\|]} = \mathcal{G}(X)$$

We can apply the same reasoning for $\mathfrak{T}$.

4.

$$\mathcal{G}(X+b) = \frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|X+b-m(X+b)\|]} = \frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|X+b-(m(X)+b-(m(X)+b)\vee 0_{\mathbb{R}^q})\|]}$$

so

$$\mathcal{G}(X+b) = \frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|X-m(X)+(m(X)+b)\vee 0_{\mathbb{R}^q}\|]}$$

Note that:

$X - m(X) \geq 0_{\mathbb{R}^q}$ and $(m(X)+b)\vee 0_{\mathbb{R}^q} \Rightarrow \|X-m(X)+(m(X)+b)\vee 0_{\mathbb{R}^q}\| \geq \|X-m(X)\|$

so

$$\mathcal{G}(X+b) \leq \mathcal{G}(X)$$

And

$\mathcal{G}(X+b) < G(X) \Leftrightarrow \|X-m(X)+(m(X)+b)\vee 0_{\mathbb{R}^q}\| > \|X-m(X)\| \Leftrightarrow (m(X)+b)\vee 0_{\mathbb{R}^q} > 0_{\mathbb{R}^q}$

$$(m(X)+b)\vee 0_{\mathbb{R}^q} > 0_{\mathbb{R}^q} \Leftrightarrow (m(X)+b)\vee 0_{\mathbb{R}^q}$$

If $(m(X)+b) = 0_{\mathbb{R}^q}$ we have $\|X-m(X)+(m(X)+b)\vee 0_{\mathbb{R}^q}\| = 0_{\mathbb{R}^q}$

Which implies $\mathcal{G}(X+b) \leq \mathcal{G}(X)$

This proof is suitable as well for $\mathfrak{T}$.

5. Let Π be a permutation matrix.

$$\mathrm{m}(\Pi \mathrm{X}) = \Pi \mathrm{m}(\mathrm{X})$$

so:

$$\|\Pi X - m(\Pi X)\| = \|\Pi X - \Pi m(X)\| = \|\Pi(X-m(X)\| = \|X-m(X)\|$$

6. Let $\beta > 0$.

$$m(\beta X) = \beta m(X)$$

so

$$\|\beta X - m(\beta X)\| = |\beta|\|X-m(X)\|$$

7. Let a=m(F) be.

$$\mathcal{G}(F) = \mathcal{G}_a(F) \text{ and } \mathfrak{T}(F) = \mathfrak{T}_a(F)$$

and we have shown $\mathcal{G}_a$ and $\mathfrak{T}_a$ are continuous. It is sufficient to consider a sequence (X_n, Y_n) such that for all n, X_n and Y_n are independent and follows the same distribution F_n such that $m(F_n) = a$ and converging to F. From the mapping theorem, we can deduce the continuity of $\mathcal{G}$ and $\mathfrak{T}$.

8. Let $\beta = (\beta_1, \ldots, \beta_q) \in \mathbb{R}^q$ be and $U = (\beta_1, X_1 \ldots, \beta_q, X_q)'$. Let $(\beta_n)_n = (\beta_1^n, \ldots, \beta_q^n)_n$ a sequence of vectors converging to β. The sequence $U_n = (\beta_1^n, X_1 \ldots, \beta_q^n, X_q)_n$ converges to U. And as $\mathcal{G}$ is continuous, the sequence $(\mathcal{G}(U_n))_n$ converges to $\mathcal{G}(U)$, in other words, the sequence $\varphi X(\beta_n))_n$ converges to $\varphi X(\beta)$. So φX is continuous.

As φX is continuous, all its restrictions are continuous.

$$\varphi^i X(0) = \mathcal{G}((X_{-i}, 0)') = \mathcal{G}(X_{-i})$$

and

$$\lim_{\beta\to+\infty} (\beta) = \lim_{\beta\to+\infty} \mathcal{G}(\beta \frac{X_{-i}}{\beta}, X_i)') = \lim_{\beta\to+\infty} \mathcal{G}((\frac{X_{-i}}{\beta}, X_i)') = \mathcal{G}((0_{\mathbb{R}^{q-1}}, X_i)') = \mathcal{G}((X_i))$$

9. An analogous reasoning to φX proves the continuity of ρX. So all its restrictions are continuous as well.

$$(\rho_X^i)'(\beta) = \frac{\beta}{C^2 \rho^i X(\beta)} (\mathbb{E}[\|X_i - \mu_i\|^2]\mathbb{E}[\|X_{-i} - a_{-i}\|^2] - \mathbb{E}[\|X_i - a_i\|^2]\mathbb{E}[\|X_{-i} - \mu_{-i}\|^2])$$

where:

$$C^2 = (\mathbb{E}[\beta^2 \|X_i - a_i\|^2 + \|X_{-i} - a_{-i}\|^2]$$

As $\beta > 0$, the sign of the derivative is exactly the sign of

$$\mathbb{E}[\|X_i - Y_i\|^2]\mathbb{E}[\|X_{-i} - a_{-i}\|^2] - \mathbb{E}[\|X_i - a_i\|^2]\mathbb{E}[\|X_{-i} - Y_{-i}\|^2]$$

This difference does not depend on β so, the sign of the derivative is constant. So the function is monotone. Note that:

$$\rho_X^i(0) = \mathfrak{T}(X_{-i})$$

and

$$\lim_{\beta\to+\infty} \rho_X^i(\beta) = \lim_{\beta\to+\infty} \mathfrak{T}(\beta \frac{X_{-i}}{\beta})) = \lim_{\beta\to+\infty} \mathfrak{T}((X_i, \frac{X_{-i}}{\beta})) = \mathfrak{T}X_i((X_i, 0_{\mathbb{R}^{q-1}})) = \mathfrak{T}(X_i)$$

10.

$$\mathfrak{T}\left(\beta_1 X_1, \ldots, \beta_q X_q\right)' = Cst \text{if and only if } \wedge_{i=1}^q \mathfrak{T}(X_i) = \vee_{i=1}^q \mathfrak{T}(X_i)$$

This is equivalent to:

$$\forall i, j \in \{1, \ldots, q\}, \mathfrak{T}(X_i) = \mathfrak{T}(X_j)$$

Corollary 1. Let $F \in \mathcal{M}^2(\mathbb{R}_+^q)$. Let $X \in \mathcal{L}^2(\mathbb{R}_+^q)$ the distribution F. We have:

1. $\forall b > 0_{\mathbb{R}^q}$,

$$\mathcal{G}(X + b) < G(X) \text{ and } \mathfrak{T}(F) < T(F)$$

2. $\forall P$ *orthogonal matrix such that* $Im(P) = \mathbb{R}_+^q$,

$$\mathcal{G}(PZ) = \mathcal{G}(X) \text{ and } \mathfrak{T}(PX) = \mathfrak{T}(X)$$

3. $\forall S$ *orthogonal matrix such that* $Im(s) = \mathbb{R}_+^q$,

$$\mathcal{G}(S(X)) = \mathcal{G}(X) \text{ and } \mathfrak{T}(S(X)) = \mathfrak{T}(X)$$

Proof: Let $F \in \mathcal{M}^2(\mathbb{R}_+^q)$. Let $X \in \mathcal{L}^2(\mathbb{R}_+^q)$ independent and following the distribution F.

1.

$$m(F) = 0_{\mathbb{R}^q}$$

so for all , $b > 0_{\mathbb{R}^q}, m(F) + b > 0_{\mathbb{R}^q}$ which implies $\mathcal{G}(X + b) < G(X)$ and $\mathfrak{T}(X + b) < G(X)$ following the proposition below.

2. As $m(F) = 0_{\mathbb{R}^q}$,

$$\mathcal{G}(X) = \frac{\mathbb{E}[\|X - Y\|]}{\mathbb{E}[\|X\|]} \text{ and } \mathfrak{T}(X) = \sqrt{\frac{\mathbb{E}[\|X - Y\|^2]}{\mathbb{E}[\|X\|^2]}}$$

and this form is obviously invariant by any orthogonal matrix and any similitude that conserve $\mathbb{R}^+$.

One illustrative example

Let's consider a distribution F on the following $\mathbb{R}^3$ subset:

$$\{(-1,3,6),(7,-2,1),(5,0,2),\left(\sqrt{3},-4,9\right),9-7,4,1)\}$$

Let X a random vector following the distribution F. We have:

$$m(F)=(-7,-4,0)$$

Let $u=(8,-1,0) and\ v=(-1,2,-3)$. We have:

$$a+m(F)=(1,-5,0), and\ b+m(F)=(-8,-2,0)$$

In $a+m(F)$, there exists at least one non negative coordinate contrarily in b + m(F). So, from the proposition, we can directly write:

$$\mathcal{G}(X+u)<G(F)\ and\ \mathfrak{T}(X+u)<T(F)$$

but

$$\mathcal{G}(X+v)<G(F)\ and\ \mathfrak{T}(X+v)=\mathfrak{T}(F)$$

3.1 Particular case of distributions on $\mathcal{M}^1\left(\mathbb{R}_+^q\right)$

In the case F is a distribution on $\mathcal{M}^2\left(\mathbb{R}_+^q\right)$, we have $m(F)=0_{\mathbb{R}^q}$. We consider in this case the definitions of inequality index take the following forms. Let $F\in\mathcal{M}^2\left(\mathbb{R}_+^q\right)$ be a distribution on $\mathbb{R}_+^q$. We have:

$$\mathcal{G}(F)=\frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|X\|]}$$

and

$$\mathfrak{T}(F)=\sqrt{\frac{\mathbb{E}[\|X-\mathbb{E}(X)\|^2]}{\mathbb{E}[\|X\|^2]}}$$

where X and Y are two independent random vectors in $\mathcal{L}^2(\mathbb{R}^q)$ following the distribution F.

Extension of the Gini Index to multivariate distributions

This definition of G is a multivariate extension of the Gini Index on $\mathcal{L}^2(\mathbb{R}_+^q)$. In fact, we have:

$$\forall x\in\mathbb{R}^q, \|x\|=\sqrt{\sum_{i=1}^{q}|x_i|^2}$$

This index $\mathcal{G}$ becomes:

$$\mathcal{G}_1(F)=\frac{\mathbb{E}[\|X-Y\|]}{2\mathbb{E}[\|X\|]}=\frac{\mathbb{E}\left[\sqrt{\sum_{i=1}^{q}(X_i-Y_i)^2}\right]}{2\mathbb{E}\left[\sqrt{\sum_{i=1}^{q}|X_i|^2}\right]}$$

where $X=(X_1,\dots,X_q)$ and $Y=(Y_1,\dots.,Y_q)$ are two independent random vectors in $\mathcal{L}^1(\mathbb{R}^q)$ and that have the same distribution F.

Particularly, in $\mathcal{L}^1(\mathbb{R}_+)$ with q = 1 the index is:

$$\mathcal{G}_1(F)=\frac{\mathbb{E}[|X-Y|]}{2.\mathbb{E}[X]}$$

and this is exactly the definition of the Gini index.

An alternative to the Pietra Index to multivariate distributions

The definition of I extends the Pietra index to the multivariate case on $\mathcal{L}^1(\mathbb{R}_+^q)$. In fact, for any distribution $F\in\mathcal{L}^1(\mathbb{R}_+^q)$, we have:

$$\mathfrak{T}(F)=\sqrt{\frac{\mathbb{E}[\|X-\mathbb{E}(X)\|^2]}{\mathbb{E}[\|X\|^2]}}=\sqrt{\frac{\mathbb{E}[\sum_{i=1}^{q}|X_i-\mathbb{E}(X_i)|^2]}{\mathbb{E}\left[\sum_{i=1}^{q}|X_i|^2\right]}}$$

where X is a random vector following the distribution F. When we reduce the dimension q to 1, X is then a non-negative random variable and the index becomes:

$$\mathfrak{I}(\mathrm{F}) = \sqrt{\frac{\mathbb{E}[|X - \mathbb{E}(X)|^2]}{\mathbb{E}[X]^2}} = \sqrt{1 - \frac{|\mathbb{E}(X)|^2}{\mathbb{E}[X^2]}}$$

The Pietra index is:

$$\frac{\mathbb{E}[|X - \mathbb{E}(X)|]}{2\mathbb{E}[X]}$$

3.2 Inequality in discrete distributions with a finite support space

Let's consider a probability distribution $F = (\alpha_1, \dots, \alpha_n \ \in [0,1]^n$ on the vectors subset $\Omega(F) = \{x_1, \dots, x_n\} \subset R^q$ such that each vector x_i is taken with the probability α_i. We denote:

$$\forall i \in \{1, \dots, n\} = (x_{i1}, \dots, x_{iq})$$

and

$$\bar{x} = \sum_{i=1}^{n} \alpha_i \, x_i$$

In this particular case, we have:

$$m(F) = \left(\min_{i \in \{1,\dots,n\}} \{0, x_{i1}\}, \dots, \min_{i \in \{1,\dots,n\}} \{0, x_{iq}\} \right)$$

The inequality indexes of inequality in the distribution F are given by:

$$\mathcal{G}(F) = \frac{\sum_{i=1}^{n} \sum_{i'=1}^{n} \alpha_i \alpha_{i'} \|x_i - x_{i'}\|}{2 \sum_{i=1}^{n} \alpha_i \, \|x_i - m(F)\|}$$

and

$$\mathfrak{I}(F) = \sqrt{\frac{\sum_{i=1}^{n} \alpha_i \|x_i - \bar{x}\|^2}{\sum_{i=1}^{n} \alpha_i \|x_i - m(F)\|^2}}$$

3.3 Inequality in partially discrete distributions

Let F be a distribution on $\mathcal{L}^1(\mathbb{R}^q)$ Let X be a random vector with distribution F. We assume $X = (U, V) \in \mathbb{R}^m \times \mathbb{R}^{q-m}$where U is distributed on a discrete subset $\{u_1, \dots, u_n\} \subset \mathbb{R}^m$with a probability distribution $\prod = (\alpha_1, \dots, \alpha_n \in [0,1]^n$ such that P(U $= u_i = \alpha_i$and let $H \in L^1(\mathbb{R}^{q-m})$be the conditional distribution of. V with respect to U.

$$m(\Pi) = \left(\min_{1 \le i \le n} \{0, u_{i1}\}, \dots, \min_{1 \le i \le n} \{0, u_{im}\} \right) \in \mathbb{R}^m$$

We assume $m(H) \in \mathbb{R}^{q-m}$ The inequality indexes in the distribution F are:

$$\mathcal{G}(F) = \frac{\sum_{i=1}^{n} \sum_{i'=1}^{n} \alpha_i \alpha_{i'} \|u_i - u_{i'}\| + \mathbb{E}(\|V - V'\| | \{U = u_i, U' = u_{i'}\})|}{2 \sum_{i=1}^{n} \alpha_i \, \|u_i - m(\Pi)\| + \mathbb{E}(\|V - m(H)\|)]}$$

where $X' = (U', V') \subset \mathbb{R}^m \times \mathbb{R}^{q-m}$ is independent from X = (U, V) and also follows the distribution F.

$$\mathfrak{I}(F) = \sqrt{\frac{\sum_{i'=1}^{n} \alpha_i \|u_i - u_{i'}\|^2 + \mathbb{E}(\|V - \mathbb{E}(V|U = u_i)\|^2 | U = u_i)|}{\sum_{i=1}^{n} \alpha_i \, [\|u_i - m(\Pi)\|^2 + \mathbb{E}(\|V - m(H)\|^2)]}}$$

3.4 Intrinsic Inequality Index and Inequality Angle

In this section, we propose an approach of inequality measure that we will call the "intrinsic" inequality in the distribution of random vectors belonging to $\mathcal{L}^1(\mathbb{R}^q)$. We use the term "intrinsic" to express the invariance of the index by any similitude. The interest for a such property is the fact that similitude conserves the shape of elements which it is applied to and conserves the ratio of distances between elements. The conservation of ratio of distances between the values in the distribution should somehow conserves the inequality between them.

The classical measures of inequality are generally classified into two categories: relative inequality indexes are invariant by translation and absolute inequality indexes are ratio scale invariant. We are going to propose here an index that will be invariant by similitude such as translation, rotation, symmetry, homothety. The homothety invariance implies real scale invariance. We will also study the behavior of the index with the vector scaling.

We introduce the notion of similar vectors.

Definition 3.6. Let x and y be two vectors in $\mathbb{R}^q$. The vector y is said to be similar to vector x if there exists a similitude $f: \mathbb{R}^q \to \mathbb{R}^q$ such that:

$$y = f(x)$$

And we will write:

$$ySx$$

Note that the relation S is a relation of equivalence:

1. $\forall x \in \mathbb{R}^q, xSx$
2. $\forall x, y \in \mathbb{R}^q, (ySx) \Leftrightarrow (xSy)$
3. $\forall x, y, z \in \mathbb{R}^q, (xSy, ySz) \Leftrightarrow (xSz)$

If f is a similitude, then there exists $\lambda > 0$ such that for any vector x, we have:

$$\|f(x)\| = \lambda \|x\|$$

We propose the following definitions.

Definition 3.7 (Intrinsic Inequality Index and Angle Inequality). Let $F \in \mathcal{M}^1(\mathbb{R}_+^q)$. Let X and Y be two independent random vectors following the distribution F. We define the intrinsic inequality of F as follows:

$$R(F) = \frac{\mathbb{E}[\|X - Y\|]}{2.\mathbb{E}[\|X - \mathbb{E}(X)\|]} \text{ if } F \text{ not a one} - \text{point distribution}$$

and

$$R(F) = 0 \text{ for all } F \text{ a one-point distribution.}$$

We define the Inequality Angle as follows:

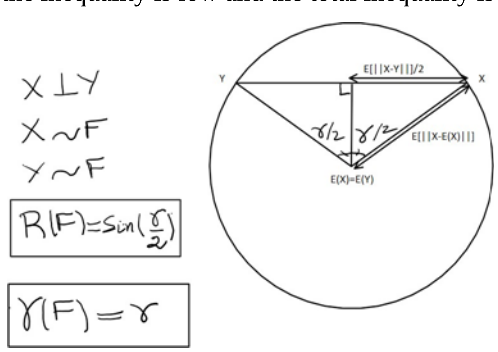
$$\gamma(F) = 2\arcsin\left(R(F)\right)$$

We can immediately give the following proposition.

The index R(F) measures how different are the values taken by X with respect to its distance to its ideal position $\mathbb{E}(X)$. The lower R(F) is, the fewer is the intrinsic inequality. The closer R(F) is to the unit, the greater is the intrinsic inequality. The inequality angle is the angle$|X - \mathbb{E}(X), Y - \mathbb{E}(X)|$.

$$0 \leq \gamma(F) \leq \pi$$

The inequality angle $\gamma(F)$ is close to 0 when the inequality is low and the total inequality is traduced by $\gamma(F) = \pi$.



Illustration

The following proposition and the corollary state some of the properties of this index.

Proposition 4. Let $F \in \mathcal{M}^1(\mathbb{R}_+^q)$, and $X \in \mathcal{L}^1(\mathbb{R}_+^q)$ following the distribution F.

We have:

1.

$$0 \leq R(F) < 1 and\ 0 \leq \gamma(F) < \pi$$

and R(F) = 0 if and only if F is a one-point distribution.

2. $\forall S: \mathbb{R}^q \to \mathbb{R}^q\ a\ similitude, R(X) = R(S(X))$

Proof.

1. Note that $R(F) = \mathcal{G}_{\mathbb{E}(X)}(F)$. This is why we have:

$$0 \leq R(F) < 1$$

and R(F) = 0 if and only if F is a one-point distribution. And $0 \leq R(F) < 1$ is equivalent to $0 \leq \gamma(F) < \pi$.

2. Let $f: \mathbb{R}^q \to \mathbb{R}^q$ be a similitude.

$$R(f(X)) = \frac{\mathbb{E}[\|f(X) - f(Y)\|]}{2\mathbb{E}[\|f(X) - \mathbb{E}[f(X)]\|]}$$

f can be decomposed on the form f = h+u where $h: \mathbb{R}^q \to \mathbb{R}^q$ is a linear application and $u \in \mathbb{R}^q$ a vector. So:

$$\mathbb{E}[f(X)] = \mathbb{E}[h(X) + u] = \mathbb{E}[h(X)] + u = h(\mathbb{E}[X\} + u = f(\mathbb{E}[X])$$

Then, we have:

$$\|f(X) - f(Y)\| = \|X - Y\|\ and\ \mathbb{E}[\|f(X) - \mathbb{E}[f(X)]\|] = \mathbb{E}[\|f(X) - f(\mathbb{E}[X])\|] = \mathbb{E}[\|X - \mathbb{E}[X]\|$$

This proves directly the invariance of R by similitude.

Corollary 2. Let $F \in \mathcal{M}^1(\mathbb{R}_+^q)$, and $X \in \mathcal{L}^1(\mathbb{R}_+^q)$ following the distribution F.

We have:

1. $\forall Is: \mathbb{R}^q \to \mathbb{R}^q\ an\ isometry, R(X) = R(Is(X))$
2. $\forall \Pi: \mathbb{R}^q \to \mathbb{R}^q\ a\ permutation\ matrix, R(X) = R(\Pi X)$
3. $\forall P\ an\ orthogonal\ matrix, R(X) = R(PX)$
4. $\forall H: \mathbb{R}^q \to \mathbb{R}^q\ an\ homothety, R(X) = R(H(X))$
5. $\forall T: \mathbb{R}^q \to \mathbb{R}^q\ a\ translation, R(X) = R(T(X))$
6. $\forall \Gamma: \mathbb{R}^q \to \mathbb{R}^q\ a\ rotation, R(X) = R(\Gamma(X))$

All these applications are similitude.

Corollary 3. Let $X \in \mathcal{L}^1(\mathbb{R}^q)$ and $Y \in \mathcal{L}^1(\mathbb{R}^q)$. If X and Y gaussian then R(X) = R(Y)

Proof. Any gaussian distribution $\mathcal{N}(0_{\mathbb{R}^q}\Sigma)$ is an image by similitude of the standard multidimensional gaussian distribution $\mathcal{N}(0_{\mathbb{R}^q}, Id_q)$.

The following proposition gives a kind of characterization of independence and identical distribution between two random vectors with the intrinsic inequality. We give one definition before the characterization.

Definition 3.8. Let $X \in \mathcal{L}^2(\mathbb{R}^q)$ with distribution $F \in \mathcal{M}^2(\mathbb{R}^q)$ such that:

$$\mathbb{E}(X) = \mu\ and\ \mathbb{v}(X) = \mathbb{E}[(X - \mu)'] = \Sigma$$

We define K(X) as the set of all vectors $Y \in \mathcal{L}^0(\mathbb{R}^q)$ with distribution F_Y such that:

$$\mathbb{E}[Y] = \mu, \mathbb{E}[\|Y - \mu\|] = \mathbb{E}[\|X - \mu\|], \mathbb{v}(Y) = \Sigma, \mathbb{E}[(X - \mu)'] = 0_{\mathbb{R}^q}$$

and

$$R(F_Y) = R(F) = \frac{\mathbb{E}[\|X - Y\|]}{2\mathbb{E}[\|X - \mu\|]}$$

Proposition 5. Let $X \in \mathcal{L}^2(\mathbb{R}^q)$ following a distribution $F \in \mathcal{M}^2(\mathbb{R}^q)$, and let $Y \in \mathcal{L}^0(\mathbb{R}^q)$ be:

1. $Y \in \mathcal{K}(X) \Leftrightarrow X \in \mathcal{K}(Y)$
2. $(X \perp Y, Y \sim F) \Rightarrow \big(Y \in \mathcal{K}(X)\big)$
3. If $Y \in \mathcal{K}(X)$ then Y can be written on the form:

$$Y = X + 2.R(F).\mathbb{E}[\|X-\mu\|].U$$

such that:

$$\mathbb{E}(U) = 0_{\mathbb{R}^q}, \mathbb{E}[\|U\|] = 1, \mathbb{V}(U) = \Gamma = \frac{1}{2.R^2(F)(\mathbb{E}[\|X-\mu\|]}\Sigma$$

and

$$\mathbb{E}\left(XU'\right) = R(F)\mathbb{E}[\|X-\mu\|\Gamma = \frac{1}{2.R^2(F)\mathbb{E}[\|X-\mu\|}\Sigma$$

4. If X and Y gaussian then

$$Y \in K(X) \Longrightarrow Y \sim F$$

Proof.

1. The definition of K is completely symmetric for X and Y.
2. If X and Y are independent and have the same distribution, then they have the same expectation, variance matrix, R, the same expected distance to their common expectation and they are non-correlated.
3. Let $X \in \mathcal{L}^1(\mathbb{R}^q)$ following a distribution F such that $E(X) = \mu$ and $\mathbb{V}(X) = \Sigma$. Let $Y \in K(X)$. We can write:

$$\begin{aligned} Y &= X + (Y - X) \\ &= X + \mathbb{E}[\|Y-X\|]\frac{Y-X}{\mathbb{E}[\|Y-X\|]} \\ &= X + 2R(F)\mathbb{E}[\|X-\mu\|]U \end{aligned}$$

Where:

$$U = \frac{Y-X}{\mathbb{E}[\|Y-X\|]}$$

By computation, we can easily remark that:

$$\mathbb{E}(U) = 0 \ and \ \mathbb{E}(\|U\|) = 1$$

as $\mathbb{E}(X) = \mathbb{E}(Y)$.

$$\begin{aligned} \mathbb{V}(U) &= \mathbb{E}[UU'] \\ &= \mathbb{V}\left[\frac{Y-X}{2R(F)\mathbb{E}[\|X-\mu\|]}\right] \\ &= \frac{\mathbb{V}(X) - 2\mathbb{E}[X-\mu(Y-\mu)'] + \mathbb{V}(Y)}{4R^2(F)\mathbb{E}^2[\|X-\mu\|]} \end{aligned}$$

$$\mathbb{V}(U) = \frac{\Sigma}{2R^2(F)\mathbb{E}^2[\|X-\mu\|]}$$

$$\begin{aligned} \mathbb{E}[XU'] &= \frac{1}{2R(F)\mathbb{E}[\|X-\mu\|]}\mathbb{E}[X(Y-X)'] \\ &= \frac{1}{2R(F)\mathbb{E}[\|X-\mu\|]}\mathbb{E}[X(Y-\mu)' - X(X-\mu)'] \\ &= \frac{1}{2R(F)\mathbb{E}[\|X-\mu\|]}\mathbb{E}[-X(X-\mu)'] \\ &= \frac{1}{2R(F)\mathbb{E}[\|X-\mu\|]}\Sigma \end{aligned}$$

From the proposition, we can deduce particularly that the intrinsic inequality is translation invariant and scale invariant. So, for example, we have:

$$\forall(\mu,\Sigma) \in \mathbb{R}^q \times \mathcal{M}_q(\mathbb{R}), R\big(\mathcal{N}(\mu,\Sigma)\big) = \mathrm{R}\big(0_{\mathbb{R}^q}, Id_q\big)$$

where Σ is a positive matrix.

Intrinsic Inequality and Angle Inequality of some classic distributions

We simulate 1000 iid variables from each of the following distributions: N(0; 1), E(1), $(N(0;1))^2$, P(1).

Distribution	$\mathcal{N}(0,1)$	$\varepsilon(1)$	$\mathcal{N}(0,1)^2$	$\mathcal{P}(1)$
R	0.71619	0.68137	0.65562	0.6867
$\gamma = 2\arcsin(R)$	$0.50823\,\pi$ rad	$0.47723\,\pi$ rad	$0.45519\,\pi$ rad	$0.48190\,\pi$ rad

We can remark that the Chisquare distribution has the lowest intrinsic inequality and the lowest angle inequality among the considered distributions, and the standard gaussian distribution has the highest intrinsic inequality.

4. Some Empirical Applications

4.1 Inequality in simulated distributions

Let consider the following observed $\mathbb{R}^3$ vectors on a population of size n = 10

$$X = \begin{pmatrix} -1 & 2 & 0 \\ 3 & 1 & -2 \\ 4 & 0 & 3 \\ 7 & -5 & 3 \\ -2 & 4 & 0 \\ 5 & 1 & 3 \\ -4 & 7 & 1 \\ 1 & 0 & 4 \end{pmatrix}, Y = \begin{pmatrix} -5 & 3 & 1 \\ -2 & 4 & -5 \\ 1 & -3 & 1 \\ 9 & -1 & 4 \\ -4 & 5 & 1 \\ 9 & 2 & 5 \\ -1 & 6 & 2 \\ -2 & 3 & 5 \end{pmatrix}$$

We have:

Distribution	$\mathcal{G}$	$\mathfrak{I}$	R	γ=2arcsin(R)
X	0.3161782	0.4949579	0.664586	0.462784πrad
Y	0.3269362	0.5071525	0.643998	0.445451πrad

We remark that the inequality indexes $\mathcal{G}$ and $\mathfrak{I}$ of X are respectively lower than the ones of Y. There is less inequality in the distribution of X than in the distribution of Y. However, the intrinsic inequality R is lower for Y than for X.

4.2 Inequality of distribution in Iris types

We consider the data Iris from Fisher (1936). This data is relative to three types of Iris that are: Iris Setosa, Iris Versicolor and Iris Virginica. For each type of Iris, it is observed on 50 individuals, the sepal length, the sepal width, the petal length and the petal width. Assuming that each type of Iris is characterized by the vector with coordinates the sepal length, the sepal width, the petal length and the petal width, we are interested in measuring the inequality in the distribution of each type of Iris. For that we compute some indexes $\mathcal{G}$ and $\mathfrak{I}$ and we obtain the following results.

	Iris Setosa	Iris Versicolor	Iris Virginica
$\mathcal{G}$	0.0546083	0.06159254	0.06168767
$\mathfrak{I}$	0.08779544	0.09825492	0.09943826
R	0.7088158	0.691367	0.7037652

We can remark that, considering the inequality index G and I, the inequality of the Iris Versicolor is between the inequality of the Iris Setosa and the inequality of the Iris Virginica. This confirms the idea that the Iris Versicolor

is an hybrid of the Iris Setosa and Iris Virginica. The intrinsic inequality is actually the inequality of the centred vector. The intrinsic inequality of the Iris Versicolor is lower than the intrinsic inequality of Iris Setosa and Iris Virginica. This means that when we center the vectors, the inequality of the Iris Versicolor is lower than the two others.

For each type of iris, the petal width has the highest inequality indexes G and I. The following table gives the inequality indexes of the different characteristics observed on the Iris.

	Sepal Length	Sepal Width	Petal Length	Petal Width
$\mathcal{G}$	0.08042365	0.07915104	0.2614854	0.35927
$\mathfrak{I}$	0.117665	0.1101555	0.4158453	0.5487493
R	0.6834971	0.7185389	0.6288046	0.6547069

We have:

$$\text{Sepal Width} \succ \text{Sepal Length} \succ \text{Petal Length} \succ \text{Petal Width}$$

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